Introduction to Artificial Intelligence

First-order Logic

(Logic, Deduction, Knowledge Representation)

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Outline

- Why first-order logic?
- Syntax and semantics of first-order logic
- Fun with sentences
- Wumpus world in first-order logic

Pros and Cons of Propositional Logic

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pieces of syntax correspond to facts

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- ☹ **Propositional logic has very limited expressive power**
  (unlike natural language)

  Example:
  Cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First-order Logic

Propositional logic

Assumes that the world contains facts

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- **Objects**
  people, houses, numbers, theories, Donald Duck, colors, centuries, . . .

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  red, round, prime, multistoried, ...
  brother of, bigger than, part of, has color, occurred after, owns, ...

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  red, round, prime, multistoried, ...
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- **Functions**
  $+$, middle of, father of, one more than, beginning of, ...

Syntax of First-order Logic: Basic Elements

Symbols

Constants $KingJohn,\ 2,\ Koblenz,\ C, \ldots$

Predicates $Brother,\ >,\ =,\ \ldots$

Functions $Sqrt,\ LeftLegOf,\ \ldots$

Variables $x,\ y,\ a,\ b,\ \ldots$

Connectives $\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$

Quantifiers $\forall \quad \exists$

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Predicates	$Brother,\ >,\ =,\ \ldots$
Functions	$Sqrt,\ LeftLegOf,\ \ldots$
Variables	$x,\ y,\ a,\ b,\ \ldots$
Connectives	$\land \quad \lor \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Quantifiers	$\forall \quad \exists$

Note

The equality predicate is always in the vocabulary

It is written in infix notation

Syntax of First-order Logic: Atomic Sentences

Atomic sentence

$$\textit{predicate}\ (\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$$

or

$$\textit{term}_1 = \textit{term}_2$$

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or

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Term

$\textit{function}\ (\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$

or

constant

or

variable

Syntax of First-order Logic: Atomic Sentences

Example

$$Brother\,(\,KingJohn,\ RichardTheLionheart\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{Brother}_{\textbf{predicate}}\,(\,\underbrace{KingJohn}_{\textbf{constant}},\ \underbrace{RichardTheLionheart}_{\textbf{constant}}\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{Brother}_{\text{predicate}}\,(\,\underbrace{\underbrace{KingJohn}_{\text{constant}}}_{\text{term}},\;\underbrace{\underbrace{RichardTheLionheart}_{\text{constant}}}_{\text{term}}\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\underbrace{\mathit{Brother}}_{\text{predicate}} \left(\underbrace{\underbrace{\mathit{KingJohn}}_{\text{constant}}}_{\text{term}}, \underbrace{\underbrace{\mathit{RichardTheLionheart}}_{\text{constant}}}_{\text{term}} \right)}_{\text{atomic sentence}}$$

Syntax of First-order Logic: Atomic Sentences

Example

$$> \quad (\, Length(LeftLegOf(Richard)),\ Length(LeftLegOf(KingJohn))\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{>}_{\text{predicate}} \left(\underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{Richard}_{\text{constant}})),\ \underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{KingJohn}_{\text{constant}}))\right)$$

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Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \land S_2 \qquad S_1 \lor S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

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Example

$$Sibling(\,KingJohn, Richard\,) \quad \Rightarrow \quad Sibling(\,Richard, KingJohn\,)$$

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Semantics in First-order Logic

Models of first-order logic

Sentences are true or false with respect to models, which consist of

- **a domain (also called universe)**
- **an interpretation**

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Domain

A non-empty (finite or infinite) set of arbitrary elements

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Domain

A non-empty (finite or infinite) set of arbitrary elements

Interpretation

Assigns to each

– constant symbol: a domain element

– predicate symbol: a relation on the domain (of appropriate arity)

– function symbol: a function on the domain (of appropriate arity)

Semantics in First-order Logic

Definition

An atomic sentence

$$\textit{predicate}\ (\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$$

is true in a certain model (that consists of a domain and an interpretation)

iff

the domain elements that are the interpretations of $term_1, \ldots, term_n$
are in the relation that is the interpretation of *predicate*

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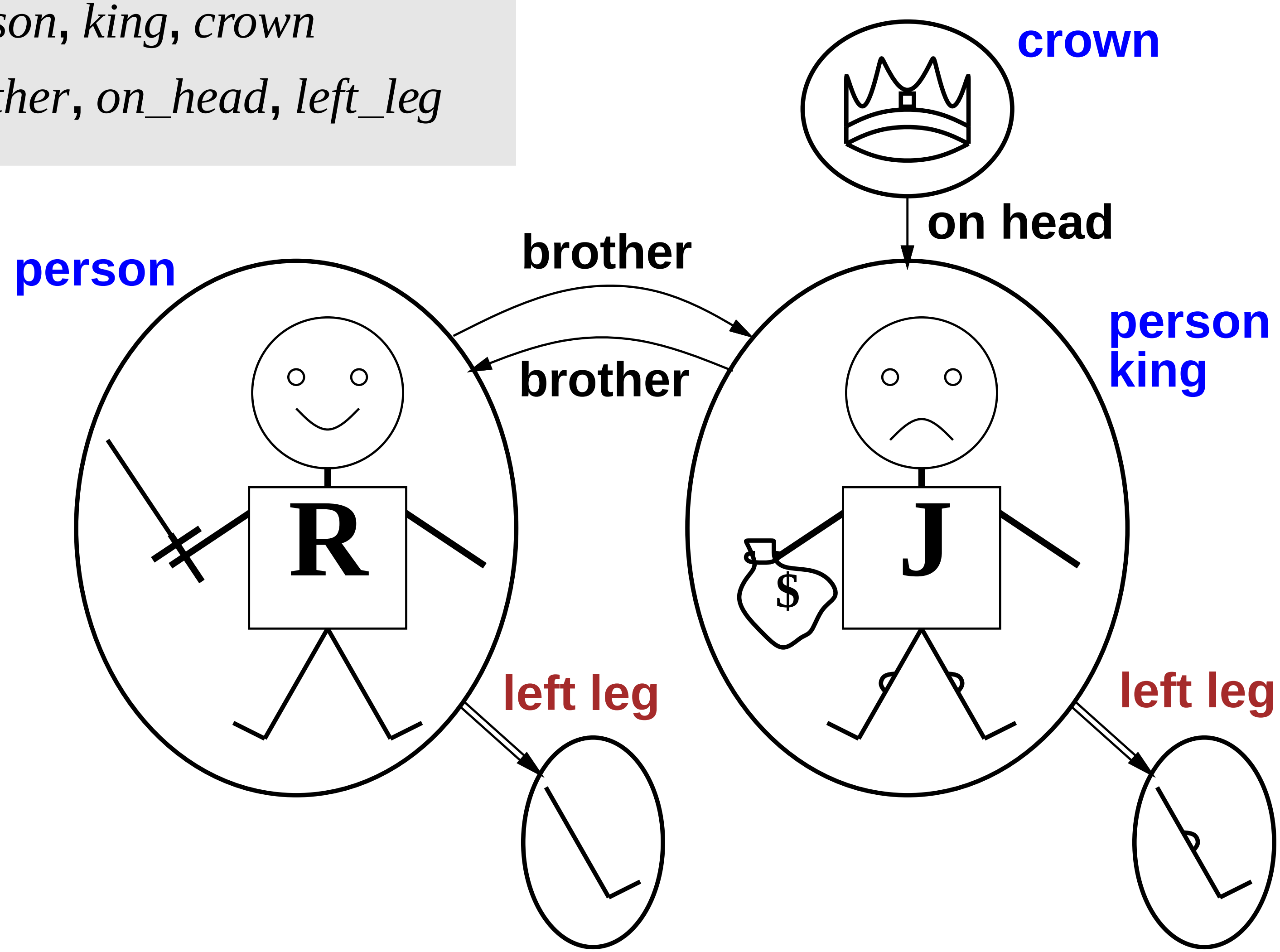
The truth value of a complex sentence in a model
is computed from the truth-values of its atomic sub-sentences
in the same way as in propositional logic

Models for First-order Logic: Example

Constants: *KingJohn*, *Richard*

Predicates: *person*, *king*, *crown*

Functions: *brother*, *on_head*, *left_leg*



Universal Quantification: Syntax

Syntax

$\forall$ *variables sentence*

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Syntax

$\forall$ *variables sentence*

Example

“Everyone studying in Koblenz is smart:

$$\underbrace{\forall \quad x}_{\text{variables}} \quad \underbrace{(StudiesAt(x, Koblenz) \Rightarrow Smart(x))}_{\text{sentence}}$$

Universal Quantification: Semantics

Semantics

$\forall x P$ **is true in a model**

iff

for all domain elements d **in the model:**
P **is true in the model when** x **is interpreted by** d

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Intuition

$\forall x P$ **is roughly equivalent to the conjunction of all instances of** P

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Intuition

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Example $\forall x\, StudiesAt(x, Koblenz) \Rightarrow Smart(x)$ **equivalent to:**

$$
\begin{array}{ll}
 & StudiesAt(KingJohn, Koblenz) \Rightarrow Smart(KingJohn) \\
\wedge & StudiesAt(Richard, Koblenz) \Rightarrow Smart(Richard) \\
\wedge & StudiesAt(Koblenz, Koblenz) \Rightarrow Smart(Koblenz) \\
\wedge & \ldots
\end{array}
$$

A Common Mistake to Avoid

Note

$\Rightarrow$ **is the main connective with** $\forall$

Common mistake

Using $\wedge$ **as the main connective with** $\forall$

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Example

Correct: $\forall x\,(StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

"Everyone who studies at Koblenz is smart"

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Example

Correct: $\forall x \, (StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

"Everyone who studies at Koblenz is smart"

Wrong: $\forall x \, (StudiesAt(x, Koblenz) \wedge Smart(x))$

"Everyone studies at Koblenz and is smart", i.e.,

"Everyone studies at Koblenz and everyone is smart"

Existential Quantification: Syntax

Syntax

$\exists$ *variables sentence*

Existential Quantification: Syntax

Syntax

$\exists$ *variables sentence*

Example

"Someone studying in Landau is smart:

$$\underbrace{\exists \quad x}_{\text{variables}} \quad \underbrace{(StudiesAt(x, Landau) \wedge Smart(x))}_{\text{sentence}}$$

Existential Quantification: Semantics

Semantics

$\exists x P$ is true in a model

iff

there is a domain element d in the model such that:
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Example $\exists\, x\, StudiesAt(x, Landau) \wedge Smart(x)$ **equivalent to:**

$$
\begin{array}{ll}
 & StudiesAt(KingJohn, Landau) \wedge Smart(KingJohn) \\
\vee & StudiesAt(Richard, Landau) \wedge Smart(Richard) \\
\vee & StudiesAt(Landau, Landau) \wedge Smart(Landau) \\
\vee & \ldots
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Example

Correct: $\exists x\, (StudiesAt(x, Landau) \wedge Smart(x))$

"There is someone who studies at Landau and is smart"

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$\wedge$ **is the main connective with** $\exists$

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Example

Correct: $\exists x\,(StudiesAt(x, Landau) \wedge Smart(x))$

"There is someone who studies at Landau and is smart"

Wrong: $\exists x\,(StudiesAt(x, Landau) \Rightarrow Smart(x))$

"There is someone who, if he/she studies at Landau, is smart"

This is true if there is anyone not studying at Landau

Properties of Quantifiers

Quantifiers of same type commute

$\forall x \forall y$	**is the same as**	$\forall y \forall x$
$\exists x \exists y$	**is the same as**	$\exists y \exists x$

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\exists x \forall y Loves(x,y)$

"There is a person who loves everyone in the world"

$\forall y \exists x Loves(x,y)$

"Everyone in the world is loved by at least one person"

(Both hopefully true but different)

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\forall x \exists y Mother(x,y)$

"Everyone has a mother" **(correct)**

$\exists y \forall x Mother(x,y)$

"There is a person who is the mother of everyone" **(wrong)**

Properties of Quantifiers

Quantifier duality

$\forall x Likes(x, IceCream)$ **is the same as** $\neg\exists x \neg Likes(x, IceCream)$

$\exists x Likes(x, Broccoli)$ **is the same as** $\neg\forall x \neg Likes(x, Broccoli)$

Fun with Sentences

- **"Brothers are siblings"**

  $\forall x,y\ (Brother(x,y) \Rightarrow Sibling(x,y))$

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- **“Sibling” is symmetric**

  $\forall x,y\ (Sibling(x,y) \Leftrightarrow Sibling(y,x))$

- **“One’s mother is one’s female parent”**

  $\forall x,y\ (Mother(x,y) \Leftrightarrow (Female(x) \land Parent(x,y)))$

Fun with Sentences

- **"Brothers are siblings"**

  $\forall x,y\ (Brother(x,y) \Rightarrow Sibling(x,y))$

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- **"One's mother is one's female parent"**

  $\forall x,y\ (Mother(x,y) \Leftrightarrow (Female(x) \land Parent(x,y)))$

- **"A first cousin is a child of a parent's sibling"**

  $\forall x,y\ (FirstCousin(x,y) \Leftrightarrow \exists p, ps\ (Parent(p,x) \land Sibling(ps,p) \land Parent(ps,y)))$

Equality

Semantics

$term_1 = term_2$ **is true under a given interpretation**

if and only if

$term_1$ **and** $term_2$ **have the same interpretation**

Equality

Example

Definition of (full) sibling in terms of $Parent$

$$\begin{aligned}\forall x,y\ Sibling(x,y) \Leftrightarrow\ &(\neg(x=y) \wedge \\ &\exists m,f\ (\neg(m=f) \wedge \\ &\quad Parent(m,x) \wedge Parent(f,x) \wedge \\ &\quad Parent(m,y) \wedge Parent(f,y)))\end{aligned}$$

Properties of First-order Logic

Important notions

- **validity**
- **satisfiability**
- **unsatisfiablity**
- **entailment**

are defined for first-order logic in the same way as for propositional logic

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Calculi

There are sound and complete calculi for first-order logic (e.g. resolution)

- **Whenever** $KB \vdash \alpha$**, it is also true that** $KB \models \alpha$
- **Whenever** $KB \models \alpha$**, it is also true that** $KB \vdash \alpha$

But these calculi CANNOT decide validity, entailment, etc.

Properties of First-order Logic

In propositional logic

Validity, satisfiability, unsatisfiablity are decidable

In first-order logic

The set of valid, and the set of unsatisfiable formulas are enumerable

The set of satisfiable formulas is NOT EVEN enumerable