Normal Forms

Literal

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- **the negation of an atomic sentence**

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A disjunction of literals

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Conjunctive Normal Form (CNF)

A conjunction of disjunctions of literals,
i.e., a conjunction of clauses

Example

$$(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$$

Resolution

Inference rule

$$\frac{P_1 \vee \cdots \vee P_{i-1} \vee Q \vee P_{i+1} \vee \ldots \vee P_k \qquad\qquad R_1 \vee \cdots \vee R_{j-1} \vee \neg Q \vee R_{j+1} \vee \ldots \vee R_n}{P_1 \vee \cdots \vee P_{i-1} \vee P_{i+1} \vee \ldots \vee P_k \;\vee\; R_1 \vee \cdots \vee R_{j-1} \vee R_{j+1} \vee \ldots \vee R_n}$$

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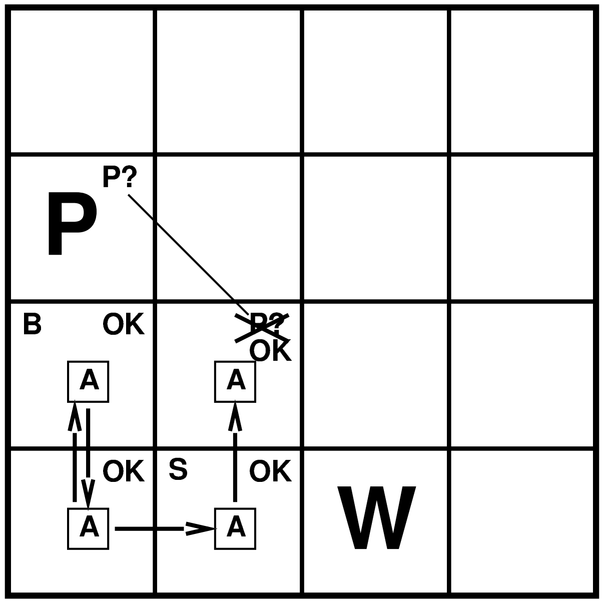
Example

$$\frac{P_{1,3} \vee P_{2,2} \qquad \neg P_{2,2}}{P_{1,3}}$$



Resolution

Correctness theorem

Resolution is sound and complete for propositional logic,

i.e., given a formula α in CNF (conjunction of clauses):

> α **is unsatisfiable**
> **iff**
> **the empty clause can be derived from α with resolution**

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$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

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$$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$

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3. Move $\neg$ inwards using de Morgan's rules (and double-negation)

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4. **Apply distributivity law (**$\vee$ **over** $\wedge$**) and flatten**

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Resolution Example

Given

$$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$$

$$\alpha = \neg P_{1,2}$$

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Derive empty clause $\square$ from $KB \wedge \neg\alpha$ in CNF

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