Uniform Variable Splitting

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The context of this work:

- free variable tableaux / sequent calculi
- classical first-order logic
- proof search
- proof theory
- relation to KeY & incremental closure

Variable independence

A free variable can occur in different context/branches.

Question: When are variable occurrences independent, i.e. when is it sound to instantiate them differently?

Goal: To label variables differently (modulo a set of equations) exactly when they are independent.

γ-rules

Free variables are introduced.

$$\frac{\Gamma, \forall x \varphi, \varphi[x/u] \vdash \Delta}{\Gamma, \forall x \varphi \vdash \Delta} \mathsf{L}\forall$$

$$\frac{\Gamma \vdash \exists x \varphi, \varphi[x/u], \Delta}{\Gamma \vdash \exists x \varphi, \Delta} \mathsf{R}\exists$$

δ-rules

Skolem functions are introduced.

$$\frac{\Gamma \vdash \varphi[x/f(\vec{u})], \Delta}{\Gamma \vdash \forall x \varphi, \Delta}\,\mathsf{R}\forall$$

$$\frac{\Gamma, \varphi[x/f(\vec{u})] \vdash \Delta}{\Gamma, \exists x \varphi \vdash \Delta}\,\mathsf{L}\exists$$

Example

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa \qquad \mathbf{Pu} \vdash Pb}{Pu \vdash Pa \wedge Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

The two occurrences of u are independent.

Example

We change the order of rule application,

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa}{\forall x Px \vdash Pa} \qquad \dfrac{\mathbf{Pv} \vdash Pb}{\forall x Px \vdash Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

and two different variables can be introduced.

Example

With sharing of variables:

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa}{\forall x Px \vdash Pa} \qquad \dfrac{\mathbf{Pu} \vdash Pb}{\forall x Px \vdash Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

The two occurrences of u are independent.

Example - variable splitting

$$\dfrac{\dfrac{\mathbf{Pu}^1 \vdash Pa \qquad \mathbf{Pu}^2 \vdash Pb}{Pu \vdash Pa \wedge Pb}}{\forall x Px \vdash \underset{1}{Pa} \wedge \underset{2}{Pb}}$$

The occurrences are labeled differently.

Example - variable splitting

$$\dfrac{\dfrac{\mathbf{Pu}^1 \vdash Pa}{(\forall x Px)^1 \vdash Pa} \qquad \dfrac{\mathbf{Pu}^2 \vdash Pb}{(\forall x Px)^2 \vdash Pb}}{\forall x Px \vdash \underset{1}{Pa} \wedge \underset{2}{Pb}}$$

The order does not matter.

β-rules and colored variables

β-rules add *indices* to *splitting sets* – example:

$$\dfrac{(\forall x Pxu)\{1,3,6,7\} \vdash \varphi \qquad (\forall x Pxu)\{1,3,6,8\} \vdash \psi}{(\forall x Pxu)\{1,3,6\} \vdash \underset{7}{\varphi} \wedge \underset{8}{\psi}}$$

Splitting sets are used to label variables, u^{1367}, u^{1368}, and unification can now be performed on the level of such *colored variables*.

Comparison to universal/local variables

$$\dfrac{\dfrac{\dfrac{\dot{P}u^1 \vdash \dot{P}a, Qa}{Pu^1 \vdash Pa \vee Qa} \qquad \dfrac{\dot{P}u^2 \vdash \dot{P}b, Qb}{Pu^2 \vdash Pb \vee Qb}}{Pu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)} \qquad \dfrac{\dfrac{\dot{Q}u^1 \vdash Pa, \dot{Q}a}{Qu^1 \vdash Pa \vee Qa} \qquad \dfrac{\dot{Q}u^2 \vdash Pb, \dot{Q}b}{Qu^2 \vdash Pb \vee Qb}}{Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)}}{\dfrac{Pu \vee Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)}{\underset{u}{\forall x}(Px \vee Qx) \vdash (Pa \underset{1}{\vee} Qa) \wedge (Pb \underset{2}{\vee} Qb)}}$$

Provability and Consistency

A substitution is *admissible* if:

1) it solves all balancing equations (e.g. $u^1 \approx u^{14}$) and

2) an induced ordering on indices is irreflexive. (This ordering gives the "right" order in which to apply rules.)

A *proof* is a derivation together with an admissible substitution closing all the leaf sequents.

Challenges