Formale Systeme II: Theorie

Dynamic Logic:
Uninterpreted and Interpreted First Order DL

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Roadmap

Overview – a family of logics

$$\begin{tikzcd}[column sep=small]
 & \text{Modal Logics} \arrow[d] & \\
 & \text{Propositional Dynamic Logic} \arrow[d] & \\
 & \text{Dynamic Logic} \arrow[dl] \arrow[dr] & \\
\text{Hybrid DL} & & \text{Java DL}
\end{tikzcd}$$

Motivation

First Order Dynamic Logic

Atomic programs are refined to **assignments**.

Example Formula

$$x_0 = x \land y_0 = y \rightarrow [x := x + y; y := x - y; x := x - y]\,\varphi$$

First Order Dynamic Logic

Inherit from FOL:

- Terms over function symbols and variables
- Predicate symbols
- Quantification over variables

Inherit from PDL

- Modalities
- Composite program constructors

Refine PDL

Unspecified atomic programs replaced by assignments $var := term$

Syntax

Syntactical material

$\Sigma = (F, P, \alpha)$ ... signature

F ... function symbols

P ... predicate symbols

$\alpha : F \cup P \to \mathbb{N}$ ... arity function

Var ... set of variables

- No atomic programs like in PDL
- Same as for FOL

Syntax

As abstract grammar:

$$
\begin{array}{rcl}
term & ::= & var \mid f(term_1, ..., term_{\alpha(f)}) \\
\\
fml & ::= & true \mid false \mid p(term_1, ..., term_{\alpha(p)}) \mid term_1 = term_2 \\
 & \mid & \neg fml \mid fml_1 \wedge fml_2 \mid fml_1 \vee fml_2 \mid fml_1 \rightarrow fml_2 \\
 & \mid & \exists var.\ fml \mid \forall var.\ fml \\
 & \mid & [prog]fml \mid \langle prog \rangle fml \\
\\
prog & ::= & var := term \\
 & \mid & var := * \\
 & \mid & prog_1 \,;\, prog_2 \mid prog_1 \cup prog_2 \mid prog^* \mid
\end{array}
$$

for $var \in Var, f \in F, p \in P$

Semantics – Kripke Structures

First Order Structure (D, I)

D … set of objects (domain) I … Interpretation
$I(f) : D^{\alpha(f)} \to D$ for function symbol $f \in F$
$I(P) \subseteq D^{\alpha(p)}$ for predicate symbol $p \in P$

Kripke Structure (S, ρ)

S … set of states $\rho : \mathtt{prog} \to S \times S$ … accessibility relation

FODL: Fixed Kripke Frame $\mathcal{K}_D = (S_D, \rho_D)$

which depends on the domain D

Semantics – Kripke Structures

The set of states $\mathcal{K}_D$ is the set of assignments of elements in the universe D to variables in Var:

$$S = Var \to D$$

For every $t \in Term_\Sigma$ we denote by

$$val_{D,I,s}(t)$$

the usual first-order evaluation of t in (D, I);
variables are interpreted via s.

Function Update Notation

Notation: for $s \in S_D$, $x \in Var$, $a \in D$

$$s[x/a](y) = \begin{cases} a & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

Semantics of Programs

Binary Relation

$\rho : \texttt{prog} \to S_D \times S_D$ assigns accessiblity to programs

$$
\begin{aligned}
\rho(x := t) &= \{(s,t) \mid t = s[x/val_{D,I,s}(t)]\} \\
\rho(x := *) &= \{(s,t) \mid \text{ex. } a \text{ with } t = s[x/a]\} \\
\rho(\pi_1 \cup \pi_2) &= \rho(\pi_1) \cup \rho(\pi_2) \\
\rho(\pi_1 \,;\, \pi_2) &= \rho(\pi_1) \,;\, \rho(\pi_2) \qquad \text{; is forward composition} \\
&= \{(s,t) \mid \text{ex. } u \in S_D \text{ with } (s,u) \in \rho(\pi_1), (u,t) \in \rho(\pi_2)\} \\
\rho(\pi^*) &= \rho(\pi)^* \qquad \text{* is refl. transitive closure} \\
&= \{(s_o, s_n) \mid \text{ex. } n \geq 0 \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ f.a. } i < n\} \\
\rho(\mathbf{?}\varphi) &= \{(s,s) \mid I, s \models \varphi\}
\end{aligned}
$$

Semantics of Formulae

$$
\begin{array}{lll}
I, s \models p(t_1, \ldots, t_n) & \text{iff} & (val_{I,s}(t_1), \ldots, val_{I,s}(t_n)) \in I(p) \\
I, s \models t_1 = t_2 & \text{iff} & val_{I,s}(t_1) = val_{I,s}(t_2) \\
I, s \models [\pi]F & \text{iff} & I, s' \models F \text{ for all } s' \text{ with } (s, s') \in \rho(\pi) \\
I, s \models \langle\pi\rangle F & \text{iff} & I, s' \models F \text{ for some } s' \text{ with } (s, s') \in \rho(\pi)
\end{array}
$$

$\models$ is homomorphic for $\neg, \wedge, \vee, \rightarrow, \forall x, \exists x$.

We write $I \models \varphi$ iff $I, s \models \varphi$ for all $s \in S$.

Basic Observation

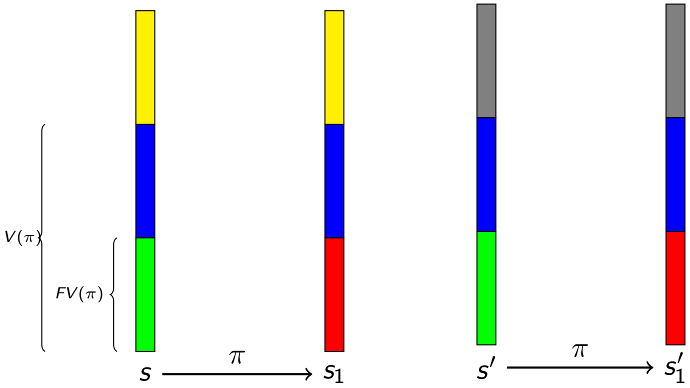
$\pi \in \texttt{prog}$ a program
$FV(\pi) = \{x \in Var \mid \text{ex. } t \text{ such that } x := t \text{ occurs in } \pi\}$
$V(\pi) = \{x \in Var \mid x \text{ occurs in } \pi\}$

1. If $(s, s_1) \in \rho(\pi)$ then $s(x) = s_1(x)$ for all $x \notin FV(\pi)$.
   i.e., program π only changes variables in $FV(\pi)$;
2. If $(s, s_1) \in \rho(\pi)$ then $(s[x/a], s_1[x/a]) \in \rho(\pi)$
   for $a \in D$, $x \notin V(\pi)$.
   i.e., variables outside $V(\pi)$ do not influence the program π;
3. more general: If $(s, s_1) \in \rho(\pi)$ and $s' \in S_D$ such that
   $s'(y) = s(y)$ for all $y \in V^\pi$ then there is s'_1 such that
   1. $(s', s'_1) \in \rho(\pi)$ and
   2. $s'_1(x) = s'(x)$ for all $x \notin V(\pi)$
   3. $s'_1(y) = s_1(y)$ for all $y \in V(\pi)$.

Basic Observation

$(s, s_1) \in \rho(\pi)$ and s' with $s'(y) = s(y)$ for all $y \in V(\pi)$
then there is s'_1 with

$$(s', s'_1) \in \rho(\pi), \qquad s'_1(x) = \begin{cases} s'(x) & \text{for all } x \notin V(\pi) \\ s_1(x) & \text{for all } x \in V(\pi) \end{cases}.$$



Interesting Tautologies

All PDL tautologies

e.g. $[\pi;\tau]\varphi \ \leftrightarrow\ [\pi][\tau]\varphi$

$$[x := t]\varphi \ \leftrightarrow\ \langle x := t\rangle\varphi$$

$$[x := *]\varphi \ \leftrightarrow\ \forall x.\varphi$$

$$\langle x := *\rangle\varphi \ \leftrightarrow\ \exists x.\varphi$$

φ a FO formula w/o quantification over x:
$[x := t]\varphi \ \leftrightarrow\ \varphi[x/t]$

$$\forall x.[\pi]\varphi \ \leftrightarrow\ [\pi]\forall x.\varphi \qquad \text{if } x \notin V(\pi)$$

Example

$$z = y \;\wedge\; \forall x.\; f(g(x)) = x \\ \quad \rightarrow \quad [(y := g(y))^*]\langle(y := f(y))^*\rangle y = z$$

$$z = y \;\wedge\; \forall x.\; f(g(x)) = x \\ \rightarrow \quad [\textbf{while } p(y) \textbf{ do } y := g(y)]\langle\textbf{while } y \neq z \textbf{ do } y := f(y)\rangle \mathit{true}$$

Indeterminism

DL programs can be indeterminstic

Sources of indeterminsm

- Non-deterministic choice $\cup$
- Non-deterministic iteration $*$
- Non-deterministic assignment $v := *$

Example for $v := *$:

$$\textbf{choose } x \textbf{ such that } p(x) \quad :\leftrightarrow \quad x := * \,; \mathbf{?}p(x)$$

Deterministic programs

Definition

A DL program $\pi \in \textit{prog}$ is called a while-program if:

1. $\cup$ occurs only within the patterns of **if**,
2. $*$ occurs only within the patterns of **while**,
3. $\textit{var} := *$ does not occur for any variable $\textit{var} \in \textit{Var}$

Reminder

$$
\begin{aligned}
\textbf{if } \varphi \textbf{ then } \alpha \textbf{ else } \beta \quad &:= \quad \big(\textbf{?}\varphi \,;\, \alpha\big) \cup \big(\textbf{?}\neg\varphi \,;\, \beta\big) \\
\textbf{while } \varphi \textbf{ do } \alpha \quad &:= \quad (\textbf{?}\varphi \,;\, \alpha)^* \,;\, \textbf{?}\neg\varphi
\end{aligned}
$$

Deterministic programs

Semantic Definition

A program $\pi \in \texttt{prog}$ is called deterministic if its accessibility relation is a partial function.

i.e., if $(s, t_1), (s, t_2) \in \rho(\pi) \implies t_1 = t_2$

Characterisation of deterministic programs

A program $\pi \in \texttt{prog}$ is deterministic iff $\langle\pi\rangle\varphi \to [\pi]\varphi$ is a tautology for every formula $\varphi \in \texttt{fml}$.

Observation

While programs are deterministic.

Deterministic programs

For determinstic programs:

$[\pi]\varphi$ means "π is **partially** correct with respect to postcondition φ"

$\langle\pi\rangle\varphi$ means "π is **totally** correct with respect to postcondition φ" (i.e. π partially correct **and** π terminates)

Moreover:

Total correctness is partial correctness plus termination:

$$\models \langle\pi\rangle\varphi \;\leftrightarrow\; [\pi]\varphi \wedge \langle\pi\rangle \mathit{true}$$

Expressiveness

Expressiveness of uninterpreted FODL

First order dynamic logic is more expressive than first order logic.

Arithmetic cannot be axiomatised in FOL

a direct implication of Gödel's Incompleteness Theorem

Arithmetic can be axiomatised in FODL

... we shall see how ...

Axiomatisation of natural arithmetic

Signature: Let Σ contain:

- constant o (the "zero")
- unary function s (the "successor")

Goal

Define a FODL formula $\varphi_{\mathbb{N}}$ over Σ s.t.
$D, I \models \varphi_{\mathbb{N}} \quad \text{iff} \quad (D, I(n), I(s)) \overset{\sim}{=} (\mathbb{N}, 0, +1)$

Idea:

Formalise: "Every element can be reached by a number of loop iterations from zero."

Solution:

$$\begin{array}{lll} \varphi_{\mathbb{N}} := & & \forall y. \langle x := o; (x := s(x))^* \rangle x = y \\ & \wedge & \forall x, y.\ ((s(x) = s(y) \to x = y) \wedge \neg s(x) = o) \end{array}$$

Interpreted Dynamic Logic

Fix the first order structure and domain.

In particular: consider

$$\Sigma_{\mathcal{N}} = (\{0, 1, -1, ..., +, *\}, \{<\}) \text{ and } \mathcal{N} = (\mathbb{N}, I_{\mathcal{N}})$$

s.t. $I_{\mathcal{N}}$ interprets the symbols "as expected".

Examples

Karlsruhe Institute of Technology

Valid formulas:

- $3 < 5, x < x + 2, 0 * x = 0$
- $(p(0) \land \forall x.(p(x) \to p(x+1))) \to \forall x.p(x)$
- $\neg \exists x(0 < x \land x < 1)$
- $[y := x \,;\, (a := * \,;\, x := x + a)^*]x \geq y$
- $x_0 = x \land y_0 = y$
  $\to [x := x + y; y := x - y; x := x - y]\, x = y_0 \land y = x_0$

Relative Completeness and Calculi

Preliminaries

Encoding sequences (*Gödel*, $\sim$1930)

There exists a first-order definable function $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with:
For every $n \in \mathbb{N}$ and every sequence $c_1, \ldots, c_n \in \mathbb{N}^*$ there exists some c such that $\beta(c, i) = c_i$ for $i = 0, \ldots n$.

c is called the *Gödel number* for $c_1, \ldots, c_n$.
Notation: $c = \ulcorner c_1, \ldots, c_n \urcorner$

Example encoding:

$$\ulcorner c_1, \ldots, c_n \urcorner := 2^{c_1+1} \cdot 3^{c_2+1} \cdot 5^{c_3+1} \cdot \ldots \cdot p_n^{1+c_n}$$

$$\beta(c, i) = k \quad \Leftrightarrow \quad p_i^{k+1} \mid c \;\wedge\; p_i^{k+2} \nmid c$$

Example: $\ulcorner 2, 0, 1 \urcorner = 2^3 \cdot 3^1 \cdot 5^2 = 600$

Comparing logics

- **Uninterpreted FODL is more expressive than FOL.**
  There exists a FODL formula such that no FOL formula has the same models.

- **Is FODL over $\mathcal{N}$ more expressive than FOL over $\mathcal{N}$?**
  How can the compare expressiveness with a fixed interpretation?

Relative Completeness

Let L be a logic.
Let $T \subseteq Fml_L$ be a set of formulas (a *theory*).

Oracle

Function $O_T : Fml_L \to \{true, false\}$ with $\varphi \in T \Leftrightarrow O(\varphi) = true$ is called an **oracle** for T.

Relative Completeness *(Cook, 1978)*

A logic is called complete relative to T if there exists a correct and complete calculus which may make use of oracle O_T.

Note: T (resp. O_T) may not be computable!

Relative Completeness of FODL

Let $T_{\mathcal{N}} = \{\varphi \mid \mathcal{N} \models \varphi\}$ be the set of valid statements over $\mathbb{N}$.

Theorem

FODL is complete relative to $T_{\mathcal{N}}$.

Programs as Formulas

Programs representable

Every DL program π can be represented as a formula $\kappa(\pi) \in Fml_{FOL_\mathcal{N}}$

Here: only one-variable-programs $V(\pi) = \{x\}$

(general case $\rightsquigarrow$ exercise)

Predicate $\kappa(\pi)(x, x')$ has two free variables:

1. x for the pre-state,
2. x' for the post-state.

Modelling goal:

$$s[x'/s'(x)] \models \kappa(\pi)(x, x') \iff (s, s') \in \rho(\pi)$$

Programs as Formulas (II)

$$
\begin{aligned}
\kappa(x := t)(x, x') &:= x' = t \\
\kappa(\pi_1 \cup \pi_2)(x, x') &:= \kappa(\pi_1)(x, x') \;\vee\; \kappa(\pi_2)(x, x') \\
\kappa(\pi_1 \,;\, \pi_2)(x, x') &:= \exists u.\ \kappa(\pi_1)(x, u) \wedge \kappa(\pi_2)(u, x') \\
\kappa(\mathbf{?}\varphi)(x, x') &:= \varphi(x) \wedge x = x' \\
\kappa(\pi^*) &:= \exists n. \exists \ulcorner x_1, \ldots, x_n \urcorner.\ x = x_1 \wedge x' = x_n \\
&\qquad \wedge \forall i < n.\ \kappa(\pi)(x_i, x_{i+1})
\end{aligned}
$$

Reduction of FODL$_{\mathcal{N}}$ to $FOL_{\mathcal{N}}$

Theorem

There is a function $\kappa : Fml_{FODL_{\mathcal{N}}} \to Fml_{FOL_{\mathcal{N}}}$ such that

- $\mathcal{N} \models \varphi \leftrightarrow \kappa(\varphi)$ and
- κ is computable.

Proof

by structural induction.

Interesting case:

$$\kappa([\pi]\varphi(x)) \;\leftrightarrow\; \forall x'.\, \kappa(\pi)(x, x') \to \kappa(\varphi(x'))$$

(Remainder left as exercise)

A practical calculus

Let φ be a FOL formula and π a program with only FOL tests.

Calculus

$$
\begin{array}{rcl}
[x := t]\varphi & \leadsto & \varphi[x/t] \\
[\pi_1 \,;\, \pi_2]\varphi & \leadsto & [\pi_1][\pi_2]\varphi \\
[\pi_1 \cup \pi_2]\varphi & \leadsto & [\pi_1]\varphi \wedge [\pi_2]\varphi \\
[\mathbf{?}\psi]\varphi & \leadsto & \psi \rightarrow \varphi \\
[\pi^*]\varphi & \leadsto & \quad \mathit{INV} \\
 & & \wedge\, (\forall \bar{x}.\ \mathit{INV} \rightarrow [\pi]\mathit{INV}) \\
 & & \wedge\, (\forall \bar{x}.\ \mathit{INV} \rightarrow \varphi)
\end{array}
$$

for an arbitrary formula $\mathit{INV} \in \mathit{Fml}_{\mathit{FOL}}$. $\bar{x} = \mathit{FV}(\pi)$

The calculus allows reduction of FODL formulae to FOL formulae

Weakest Precondition Calculus

Let φ be a FOL formula and π a **while** program (with FOL tests).

Calculus

$$
\begin{array}{rcl}
[x := t]\varphi & \rightsquigarrow & \varphi[x/t] \\
[\pi_1\,;\,\pi_2]\varphi & \rightsquigarrow & [\pi_1][\pi_2]\varphi \\
[\textbf{if}\ \psi\ \textbf{then}\ \pi_1\ \textbf{else}\ \pi_2]\varphi & \rightsquigarrow & (\psi \to [\pi_1]\varphi) \land (\neg\psi \to [\pi_2]\varphi) \\
[\textbf{while}\ \psi\ \textbf{do}\ \pi]\varphi & \rightsquigarrow & \quad \mathit{INV} \\
 & & \land\, (\forall \bar{x}.\ \mathit{INV} \land \psi \to [\pi]\mathit{INV}) \\
 & & \land\, (\forall \bar{x}.\ \mathit{INV} \land \neg\psi \to \varphi)
\end{array}
$$

for an arbitrary formula $\mathit{INV} \in \mathit{Fml}_{FOL}$. $\quad \bar{x} = FV(\pi)$

This is the weakest-precondition calculus (*Dijkstra*, 1975)

Notation: $wlp(\pi, \varphi) = [\pi]\varphi, \quad wp(\pi, \varphi) = \langle\pi\rangle\varphi$

Properties

Let $[\pi]\varphi \leadsto^* \psi$ be the result of applying the calculus.

1. $\models \psi \to [\pi]\varphi$
   ψ is a precondition such that φ is guaranteed to hold after π.

2. There exist loop invariants such that $\models \psi \leftrightarrow [\pi]\varphi$
   earlier defined $\kappa(\cdot)$ formulates **strongest loop invariants**
   Then ψ is the **weakest precondition**

3. If $\models \mathit{pre} \to \psi$, then also $\models \mathit{pre} \to [\pi]\varphi$
   Prove pre/post-condition contracts by applying calculus to program and postcondition and then showing implication from precondition.

Arithmetic Completeness

Axioms

All first-order formulas valid in $\mathcal{N}$

Axioms for PDL

$\langle x := t \rangle \varphi \quad \leftrightarrow \quad \varphi[x/t]$ for all first-order φ

Rules

$$\frac{F,\ F \rightarrow G}{G}$$ (modus ponens)

$$\frac{F}{[\pi]F} \qquad \frac{F}{\forall x F}$$ (generalisations)

$$\frac{\forall n(F(n+1) \rightarrow \langle \pi \rangle F(n))}{\forall n(F(n) \rightarrow \langle \pi^* \rangle F(0))}$$ for any first-order formula F (convergence)

Theorem

For **any** formula $\varphi \in Fml_{FODL}$: $\mathbb{N} \models \varphi \iff \vdash_{\mathbb{N}} \varphi$