Karlsruhe Institute of Technology

Formale Systeme II: Theorie

Separation Logic

SS 2016

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Motivation

Reminder . . . Dynamic Logic

Given: a program with a contract:

1. precondition, FOL formula *pre*
2. postcondition, FOL formula *post*
3. code, while program π

In program verification, one formally proves that

$$\mathbb{N} \models pre \rightarrow [\pi]post$$

If *pre* holds before execution of π then *post* holds after termination.

Reminder: weakest precondition calculus for DL.

The Framing Problem

Formal Software Verification

- Prove what effects a program has.

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Example (after McCarthy and Hayes, 1969)

P calls operator to ask for Q's number.

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P calls operator to ask for Q's number.

- Precondition: P has a telephone.

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Example (after McCarthy and Hayes, 1969)

P calls operator to ask for Q's number.

- Precondition: P has a telephone.
- Postcondition: P knows the number of Q

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- Prove what effects a program has.
- Prove what effects a program does *not* have.

You should not have to specify the latter explicitly.

Example (after McCarthy and Hayes, 1969)

P calls operator to ask for Q's number.

- Precondition: P has a telephone.
- Postcondition: P knows the number of Q
- **missing postcondition?**
  Postcondition: P still has a telephone.

The Framing Problem

Example in Java

```
interface Account {
  void setBalance(int);
  void getBalance();
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interface Account {
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//@ ensures \result == 100;
int f(Account account1, Account account2) {
  account1.setBalance(100);
  account2.setBalance(200);
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The Framing Problem

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interface Account {
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The Framing Problem

Specify what does *not* change

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- setBalance does not effect other accounts
- setBalance does not effect other customer objects

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```

Specify what does *not* change

- `setBalance` does not effect other accounts
- `setBalance` does not effect other customer objects
- `setBalance` does not effect any object of any classes which may be added later.

The Framing Problem

Problem statement

In program verification, the framing problem is the problem to specify and verify that the effects of a program are limited to the data structure that is being operated on.

It is a challenge for specifying user (needs to think about not-effects) and for reasoning engines (increased complexity).

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Suggested solutions:

- Ownership (Types) (Noble, Vitek and Potter 1998)
- Separation Logic (Reynolds, 1999)
- Dynamic Frames (Kassios 2006)
- ...

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Heaps and “Footprints”

Heap

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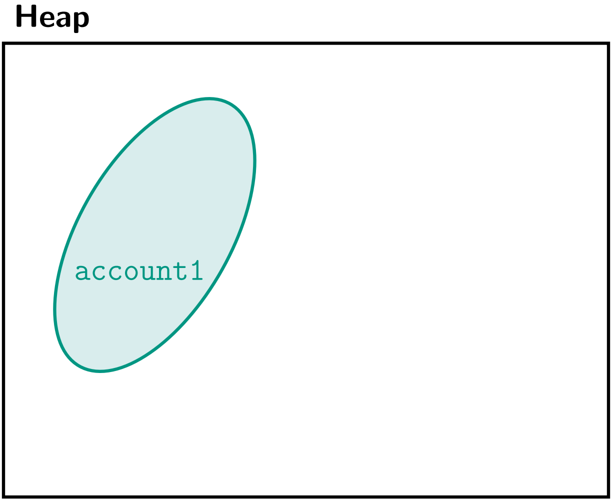

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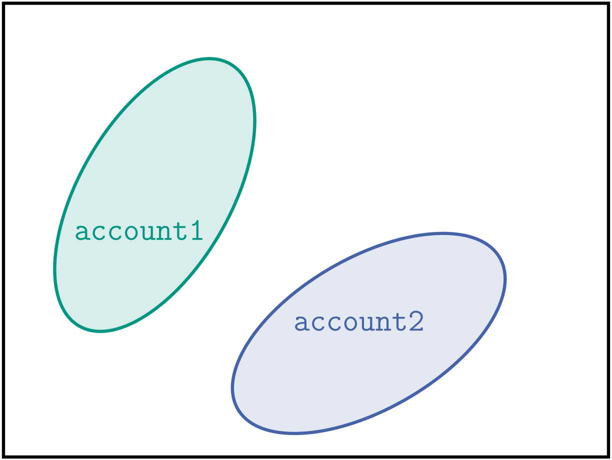

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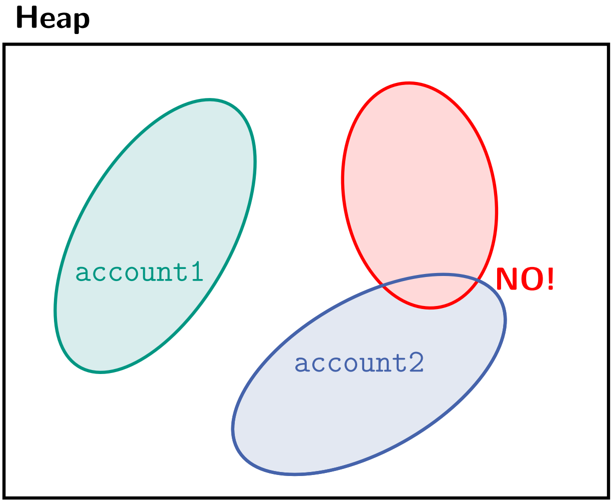

Heaps and Heaplets

Modelling assumptions

- Every memory location holds a value in $\mathbb{N}$.
- There infinitely many memory locations.

Heap and Heaplet

A **heap** is a total function modelling memory:

$$heap : \mathbb{N} \to \mathbb{N}$$

A **heaplet** is a finite partial function modelling footprints:

$$heaplet : \mathbb{N} \rightarrow\!\!\!\!\!+ \mathbb{N}$$

Partial function:

Partial function $f : A \rightarrow\!\!\!\!\!+ B$ is a function $f : D \to B$ for $D \subseteq A$.
The finite set $D = \operatorname{dom} f$ is called the domain of f.

Operations and Heaps

Disjoint union of heaplets:

$$h = h_1 \uplus h_2 \quad \text{iff} \quad \mathrm{dom}\, h_1 \cap \mathrm{dom}\, h_2 = \emptyset \text{ and } h = h_1 \cup h_2.$$

$h_1 \uplus h_2$ is always a heaplet.

(Union $\cup$ of heaplets does not always result in heaplets.)

Membership

For $(x, y) \in h$ write $h(x) = y$.

It means: Memory location x holds value y.

Empty Heap

The empty heaplet $\emptyset$ is without allocated locations.

Singletons

Heaplet with exactly one allocated location x which holds value y:

write $h = \{(x, y)\}$

Separation Logic

Separation Logic – Syntax

Terms t:

new in Separation Logic

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- FOL terms over $\mathbb{N}$ with $+, -, \cdot, 0, 1$

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- $\varphi_1 \wedge \varphi_2, \quad \varphi_1 \vee \varphi_2, \quad \varphi_1 \rightarrow \varphi_2$

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- $\varphi_1 * \varphi_2$

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Formulae φ:

- $\varphi_1 \land \varphi_2, \quad \varphi_1 \lor \varphi_2, \quad \varphi_1 \to \varphi_2$
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- **emp**

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- $\varphi_1 * \varphi_2$
- **emp**
- $t_1 \mapsto t_2$
- $\varphi_1 \mathrel{-\!\!*} \varphi_2$ (later)

new in Separation Logic

Operator Precedence

How are the implicit parentheses in
$B \twoheadrightarrow\!\!\ast\; C \wedge D \vee A \ast x \mapsto y$?

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How are the implicit parentheses in
$B \mathrel{-\!\!*} C \land D \lor A * x \mapsto y$?

Binding force:

$*$	binds like $\land$
$\mathrel{-\!\!*}$	binds like $\to$, $\lor$
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Answer:

$$\big(B \mathrel{-\!\!*} (C \land D)\big) \lor \big(A * (x \mapsto y)\big)$$

or $$B \mathrel{-\!\!*} \Big((C \land D) \lor \big(A * (x \mapsto y)\big)\Big)$$

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Add explicit parentheses when combining $\lor / \to / \twoheadrightarrow$ or $\land / *$

Separation Logic – Semantics

Structure

Fixed first order domain: $\mathbb{N}$.

Terms and formulas are evaluated over:

1. Variable assignment $\beta : \mathit{Var} \to \mathbb{N}$
2. Heaplet $h : \mathbb{N} \nrightarrow \mathbb{N}$

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Terms:

- $val_\beta(t_1 + t_2) = val_\beta(t_1) +_{\mathbb{N}} val_\beta(t_2)$, same for "$\cdot$"
- $val_\beta(x) = \beta(x)$ for variable x

Formulas in FOL:

- Operator $\beta, h \models$ is homomorphic for $\wedge, \vee, \rightarrow, \forall, \exists, <, =$.
- Example: $\beta, h \models \varphi_1 \wedge \varphi_2$ iff $\beta, h \models \varphi_1$ and $\beta, h \models \varphi_2$

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- $\beta, h \models \varphi_1 * \varphi_2$ iff there exist heaplets $h_1, h_2 : \mathbb{N} \to \mathbb{N}$ with

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  1. $h = h_1 \uplus h_2$ and
  2. $\beta, h_1 \models \varphi_1$ and
  3. $\beta, h_2 \models \varphi_2$

Separation Logic

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Connector $*$ is called **Separating Conjunction**

$A * B$ has the following intuitive semantics:

$A * B$ is true

$\iff$

A is true
and B is true
and A and B refer to
disjoint sets of memory locations.

Properties of Separation Logic

Idempotence

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  $\models \neg(7 \mapsto 3 \;\rightarrow\; 7 \mapsto 3 * 7 \mapsto 3)$

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  $\models \neg(7 \mapsto 3 * 6 \mapsto 4 \;\rightarrow\; 7 \mapsto 3)$

Caution

$\beta, h \models A \mapsto B$ means that:

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- not only $(val(A), val(B)) \in h$

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Notation sometimes: $A \hookrightarrow B \;:\leftrightarrow\; A \mapsto B * true$

Some Valid Formulas

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- $x \mapsto y \wedge x \mapsto z \;\rightarrow\; y = z$

- $P * (Q \vee R) \;\leftrightarrow\; (P * Q) \vee (P * R)$

Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?

1. $x \mapsto y * x \mapsto z$

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3. $(x \mapsto 0 \wedge y \mapsto 0) \rightarrow x = y$

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4. $(x \mapsto 0 * y \mapsto 0) \rightarrow x = y$ **UNSAT**

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3. $(x \mapsto 0 \land y \mapsto 0) \to x = y$ **VALID**
4. $(x \mapsto 0 * y \mapsto 0) \to x = y$ **UNSAT**
5. $(x \mapsto 0 * y \mapsto 0) \to \neg(x = y)$

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3. $(x \mapsto 0 \wedge y \mapsto 0) \to x = y$ **VALID**
4. $(x \mapsto 0 * y \mapsto 0) \to x = y$ **UNSAT**
5. $(x \mapsto 0 * y \mapsto 0) \to \neg(x = y)$ **VALID**

Quiz!

Karlsruhe Institute of Technology

Are the following formulas valid/satisfiable/unsatisfiable?

1. $x \mapsto y * x \mapsto z$ **UNSAT**
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5. $(x \mapsto 0 * y \mapsto 0) \rightarrow \neg(x = y)$ **VALID**
6. $(x \mapsto a \wedge y \mapsto b) \rightarrow a = b$

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Quiz!

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Are the following formulas valid/satisfiable/unsatisfiable?

1. $x \mapsto y * x \mapsto z$ **UNSAT**
2. $x \mapsto y \land x \mapsto z$ **SAT** ; true if $y = z$
3. $(x \mapsto 0 \land y \mapsto 0) \to x = y$ **VALID**
4. $(x \mapsto 0 * y \mapsto 0) \to x = y$ **UNSAT**
5. $(x \mapsto 0 * y \mapsto 0) \to \neg(x = y)$ **VALID**
6. $(x \mapsto a \land y \mapsto b) \to a = b$ **VALID**
7. $\varphi * \mathbf{emp} \to \varphi$

Quiz!

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a. $\psi * \neg\psi$ for ψ without $\mapsto$, **emp**

Quiz!

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8. $\varphi * \neg\varphi$
   a. $\psi * \neg\psi$ **UNSAT** for ψ without $\mapsto$, **emp**
   b. $x \mapsto y * \neg(x \mapsto y)$

Quiz!

Karlsruhe Institute of Technology

Are the following formulas valid/satisfiable/unsatisfiable?

1. $x \mapsto y * x \mapsto z$ **UNSAT**
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6. $(x \mapsto a \wedge y \mapsto b) \rightarrow a = b$ **VALID**
7. $\varphi * \mathbf{emp} \rightarrow \varphi$ **VALID**
8. $\varphi * \neg\varphi$
   a. $\psi * \neg\psi$ **UNSAT** for ψ without $\mapsto$, **emp**
   b. $x \mapsto y * \neg(x \mapsto y)$ **SAT**, equivalent to $x \mapsto y * \mathit{true}$

The Magic Wand

Modus Ponens for classical logic

$$\frac{A \land (A \to B)}{B}$$

The Magic Wand

Modus Ponens for classical logic

$$\frac{A \wedge (A \to B)}{B}$$

Corresponding rule for separating conjunction $*$?

The Magic Wand

Modus Ponens for classical logic

$$\frac{A \land (A \to B)}{B}$$

Corresponding rule for separating conjunction $*$?

Modus Ponens for separation logic

$$\frac{A * (A \mathrel{-\!\!*} B)}{B}$$

The **magic wand operator** $A \mathrel{-\!\!*} B$, aka **separating implication**:

$$\beta, h \models A \mathrel{-\!\!*} B$$

$$\iff$$

for all $h', h^+ : \mathbb{N} \nrightarrow \mathbb{N}$: If $h^+ = h \uplus h'$ and $h' \models A$, then $h^+ \models B$

Separating Operators

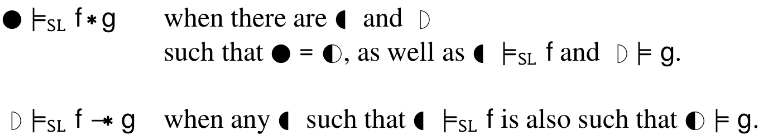


Figure 1.5: Visual representation of the semantics of separation operators

Taken from:
Separation Logic: Expressiveness, Complexity, Temporal Extension
Rémi Brochenin, PhD Thesis. 2013

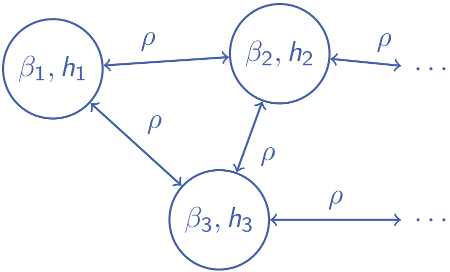
Programs and Separation Logic

Programming Language

statement	::=	`while` formula `do` statement
	|	`if` formula `then` statement `else` statement
	|	statement `;` statement
	|	var `:=` term
	|	`[`term`]` `:=` term
	|	var `:=` `[`term`]`
(later)	|	var `:=` `cons(`term`,` `...,` term`)`
(later)	|	`dispose(`var`)`

Kripke Frames with Heaps

- Every state is a pair (β, h) with $\beta : \mathit{Var} \to \mathbb{N}$ and $h : \mathbb{N} \nrightarrow \mathbb{N}$
- Kripke state transition the program semantics $\rho(st) \in S \times S$ for any statement st.



Program semantics (repetition from FODL)

Accessiblity Relation for Programs

$\rho : \texttt{statement} \to S \times S$

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$\rho : \mathtt{statement} \rightarrow S \times S$

$$\rho(\pi_1 \cup \pi_2) \quad = \quad \rho(\pi_1) \cup \rho(\pi_2)$$

Program semantics (repetition from FODL)

Accessiblity Relation for Programs

$\rho : \texttt{statement} \to S \times S$

$$\begin{aligned} \rho(\pi_1 \cup \pi_2) &= \rho(\pi_1) \cup \rho(\pi_2) \\ \rho(\pi_1 \,;\, \pi_2) &= \rho(\pi_1) \,;\, \rho(\pi_2) \qquad \text{; is forward composition} \end{aligned}$$

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\begin{aligned}
\rho(\pi_1 \cup \pi_2) &= \rho(\pi_1) \cup \rho(\pi_2) \\
\rho(\pi_1 \,;\, \pi_2) &= \rho(\pi_1) \,;\, \rho(\pi_2) \qquad \text{; is forward composition} \\
&= \{(s,t) \mid \text{ex. } u \in S \text{ with } (s,u) \in \rho(\pi_1), (u,t) \in \rho(\pi_2)\}
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\rho(\pi^*) &= \rho(\pi)^* \qquad \text{* is refl. transitive closure}
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$$
\begin{aligned}
\rho(\pi_1 \cup \pi_2) &= \rho(\pi_1) \cup \rho(\pi_2) \\
\rho(\pi_1 ; \pi_2) &= \rho(\pi_1) ; \rho(\pi_2) \qquad \text{; is forward composition} \\
&= \{(s, t) \mid \text{ex. } u \in S \text{ with } (s, u) \in \rho(\pi_1), (u, t) \in \rho(\pi_2)\} \\
\rho(\pi^*) &= \rho(\pi)^* \qquad \text{* is refl. transitive closure} \\
&= \{(s_o, s_n) \mid \text{ex. } n \geq 0 \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ f.a. } i < n\}
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\rho(\pi_1\,;\pi_2) &= \rho(\pi_1)\,;\rho(\pi_2) \qquad \text{; is forward composition} \\
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\rho(\mathbf{?}\varphi) &= \{(s,s) \mid s \models \varphi\}
\end{aligned}
$$

Reminder: IF and WHILE

$$
\begin{aligned}
\textbf{if } \varphi \textbf{ then } \alpha \textbf{ else } \beta &= (\mathbf{?}\varphi \,;\, \alpha) \cup (\mathbf{?}\neg\varphi \,;\, \beta) \\
\textbf{while } \varphi \textbf{ do } \alpha &= (\mathbf{?}\varphi \,;\, \alpha)^* \,;\, \mathbf{?}\neg\varphi
\end{aligned}
$$

Program semantics (with heap)

Accessiblity Relation for Programs

$\rho : \texttt{statement} \to S \times S$

A state $s \in S$ is a pair (β, h) with $\beta : \mathit{Var} \to \mathbb{N}$ and $h : \mathbb{N} \nrightarrow \mathbb{N}$

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A state $s \in S$ is a pair (β, h) with $\beta : Var \to \mathbb{N}$ and $h : \mathbb{N} \nrightarrow \mathbb{N}$

$$\big((\beta, h), (\beta', h')\big) \in \rho(v := t) \quad \Longleftrightarrow \quad \beta' = \beta[v/val_\beta(t)] \text{ and } h' = h$$

Program semantics (with heap)

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$$\big((\beta, h), (\beta', h')\big) \in \rho(v := [t]) \quad \Longleftrightarrow \quad \begin{array}{l} \mathit{val}_\beta(t) \in \operatorname{dom} h \text{ and } h' = h \text{ and} \\ \beta' = \beta\big[v/h[\mathit{val}_\beta(t)]\big] \end{array}$$

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$$\big((\beta, h), (\beta', h')\big) \in \rho(v := [t]) \quad \Longleftrightarrow \quad \begin{array}{l} val_\beta(t) \in \operatorname{dom} h \text{ and } h' = h \text{ and} \\ \beta' = \beta\big[v/h[val_\beta(t)]\big] \end{array}$$

$$\big((\beta, h), (\beta', h')\big) \in \rho([t] := u) \quad \Longleftrightarrow \quad \begin{array}{l} val_\beta(t) \in \operatorname{dom} h \text{ and } \beta' = \beta \text{ and} \\ h' = h[val_\beta(t)/val_\beta(u)] \end{array}$$

(Remember: $f[a/b](a) = b$ and $f[a/b](x) = f(x)$ for $x \neq a$)

Failing executions

Statement $x := [10]$ must not be executed if $10 \notin \mathrm{dom}\, h$.

State $(\beta, \emptyset)$ has no successor state in $\rho(x := [10])$.

How to distinguish between failed test $\mathbf{?}\psi$ and memory violation?

Failing executions

Statement $x := [10]$ must not be executed if $10 \notin \mathrm{dom}\, h$.

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How to distinguish between failed test $\mathbf{?}\psi$ and memory violation?

Model unallowed heap access:

$\mathit{fail} : \mathtt{statement} \to S$

$s \in \mathit{fail}(\pi)$ means: π started in s may cause memory violation.

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$$\begin{aligned} \mathit{fail}(x := t) &= \\ \mathit{fail}(\mathbf{?}\psi) &= \emptyset \end{aligned}$$

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$$\begin{aligned}
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\end{aligned}
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Failing executions

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\mathit{fail}(\pi^*) &= \rho(\pi^*) \,;\, \mathit{fail}(\pi)
\end{aligned}
$$

with $A \,;\, B = \{x \mid \text{ex } y \text{ with } (x, y) \in A \text{ and } y \in B\}$

Fail-aware modality

Remember:

$s \models [\pi]\varphi \quad \text{iff} \quad s' \models \varphi$ for all $(s, s') \in \rho(\pi)$.

Problem:

$\mathbf{emp} \to \big[[5] := 42\big]\,\mathit{false}$ is a valid formula.

New modality $[\![\cdot]\!]$

$s \models [\pi]\varphi \quad \text{iff} \quad s' \models \varphi$ for all $(s, s') \in \rho(\pi)$ and $s \notin \mathit{fail}(\pi)$

Now:

$\mathbf{emp} \to [\![[5] := 42]\!]\,\psi$ is not valid for any ψ

Dynamic Separation Logic

Valid formulas:

- $x \mapsto 5 \rightarrow [\![v := [x]\,;[x] := v + 1]\!] x \mapsto 6$
- $(\exists y.x \mapsto y) \rightarrow [\![[x] := 7]\!] x \mapsto 7$
- $x \mapsto 5 * y \mapsto 6 \;\rightarrow\; [\![[x] := 7]\!](x \mapsto 7 * y \mapsto 6)$

A Calculus for Separation Logic

Hoare Calculus

Separation Logic originally formulated as rules for a *Hoare* calculus.

A Calculus for Separation Logic

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Hoare Calculus (1969, Hoare and Floyd)

Operates on **Hoare Triples:** $\{P\}\ \pi\ \{Q\}$

A Hoare triple is valid if program π started in a state that satisfies precondition P terminates in a state which satisfies postcondition Q (it it terminates).

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Semantically the same as $P \to [\![\pi]\!]Q$.

We present the calculus using dynamic logic notation.

Calculus Rules for Sep Logic

$$x = m \;\rightarrow\; [\![x := E]\!] x = E[x{\leftarrow}m]$$

$$x = m \wedge E \mapsto n \;\rightarrow\; [\![x := [E]]\!](x = n \wedge E[x{\leftarrow}m] \mapsto n)$$

$$(\exists x. E \mapsto x) \;\rightarrow\; [\![[E] := F]\!]\; E \mapsto F$$

x, m, n distinct variables; E, F terms of $\mathbb{N}$.
$E[x{\leftarrow}F]$ is substitution: replaces all free occurrences of x in E by F.

Calculus Rules for Sep Logic

$$\frac{P \to [\![\pi_1]\!]Q \quad Q \to [\![\pi_2]\!]R}{P \to [\![\pi_1 \,;\, \pi_2]\!]R}$$

$$\frac{P \wedge C \to [\![\pi_1]\!]Q \quad P \wedge \neg C \to [\![\pi_2]\!]Q}{P \to [\![\text{if } C \text{ then } \pi_1 \text{ else } \pi_2]\!]Q}$$

$$\frac{P \wedge C \to [\![\pi]\!]P}{P \to [\![\text{while } C \text{ do } \pi]\!](P \wedge \neg C)}$$

$$\frac{P \to [\![\pi]\!]Q}{(\exists x.P) \to [\![\pi]\!](\exists x.Q)} \text{ if } x \notin \mathit{Free}(\pi)$$

(Normal rules of Hoare Calculus – nothing special for Sep Logic)

The Frame Rule

THIS IS THE KEY POINT ABOUT SEPARATION LOGIC

$$\frac{P \qquad \rightarrow [\![\pi]\!]\; Q}{P * R \rightarrow [\![\pi]\!](Q * R)}$$

$$\mathit{Modifies}(\pi) \cap \mathit{Free}(R) = \emptyset$$

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THIS IS THE KEY POINT ABOUT SEPARATION LOGIC

$$\frac{P \quad \to [\![\pi]\!]\; Q}{P * R \to [\![\pi]\!](Q * R)}$$

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Separation in Proofs

Proof $P \to [\![\pi]\!]Q$ using in P, Q the memory π refers to.
Get for free: Nothing besides these memory locations has changed.

Remember: The Framing Problem

Example in Java

```
//@ requires account1 != account2;
//@ ensures \result == 100;
int f(Account acc1, Account acc2) {
  acc1.setBalance(100);
  acc2.setBalance(200);
  return acc1.getBalance();
}
```

Rule for setBalance:

$$A \mapsto x \;\rightarrow\; [\![setBalance(A, y)]\!] A \mapsto y$$

Remember: The Framing Problem

Example in Java

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//@ requires account1 != account2;
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  return acc1.getBalance();
}
```

Rule for setBalance:

$$A \mapsto x \;\rightarrow\; [\![setBalance(A, y)]\!] A \mapsto y$$

Use Frame Rule:

$$acc2 \mapsto x \;\rightarrow\; \ldots$$

$$\ldots [\![setBalance(acc2, 200);]\!] acc2 \mapsto 200$$

Remember: The Framing Problem

Example in Java

```
//@ requires account1 != account2;
//@ ensures \result == 100;
int f(Account acc1, Account acc2) {
  acc1.setBalance(100);
  acc2.setBalance(200);
  return acc1.getBalance();
}
```

Rule for setBalance:

$$A \mapsto x \;\rightarrow\; [\![setBalance(A, y)]\!] A \mapsto y$$

Use Frame Rule:

$$acc2 \mapsto x * acc1 \mapsto 100 \;\rightarrow\; \ldots$$

$$\ldots [\![setBalance(acc2, 200);]\!] acc2 \mapsto 200 * acc1 \mapsto 100$$

On the board ...

$$(\exists v.\ \mathtt{X} \mapsto v * \mathtt{Y} \mapsto v) \;\rightarrow\; [\![\mathtt{X} := [\mathtt{X}]\,;\, \mathtt{Y} := [\mathtt{Y}]]\!]\ \mathtt{X} = \mathtt{Y}$$

Soundness of Frame Rule

$$\frac{P \to [\![\pi]\!]Q}{P * R \to [\![\pi]\!](Q * R)} \quad \text{or equivalently} \quad \frac{([\![\pi]\!]Q) * R}{[\![\pi]\!](Q * R)}$$

if $\mathit{Modifies}(\pi) \cap \mathit{Free}(R) = \emptyset$

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Premiss: trivially true, conclusion: desired implication.

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By premiss: $\beta, h_1 \models [\![\pi]\!]Q$ and $\beta, h \models ([\![\pi]\!]Q) * R$

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$\square$

Soundness of Frame Rule

$$\frac{([\![\pi]\!]Q) * R}{[\![\pi]\!](Q * R)} \text{ if } \mathit{Modifies}(\pi) \cap \mathit{Free}(R) = \emptyset$$

Proof by structural induction over π.

Case $x := t$

Let $\beta, h \models ([\![\pi]\!]Q) * R$, i.e., $\beta, h_1 \models [\![\pi]\!]Q$ and $\beta, h_2 \models R$, $h = h_1 \uplus h_2$.

x does not occur in R (by side condition):

$$\models R \;\leftrightarrow\; [\![\pi]\!]R$$

Therefore: $\beta, h_1 \models [\![\pi]\!]Q$ and $\beta, h_2 \models [\![\pi]\!]R$

After assignment: $\beta[x/val_\beta(t)], h_1 \models Q$ and $\beta[x/val_\beta(t)], h_2 \models R$

and $\beta, h \models [\![\pi]\!](Q * R)$

Soundness of Frame Rule

$$\frac{([\![\pi]\!]Q) * R}{[\![\pi]\!](Q * R)} \text{ if } \mathit{Modifies}(\pi) \cap \mathit{Free}(R) = \emptyset$$

Proof by structural induction over π.

Case $[t] := u$

Let $\beta, h \models ([\![\pi]\!]Q) * R$, i.e., $\beta, h_1 \models [\![\pi]\!]Q$ and $\beta, h_2 \models R$, $h = h_1 \uplus h_2$.

$$\begin{tikzcd}
& h_1 \models ([\![\pi]\!]Q) \arrow[r, "\pi"'] & h_{1Q} \models Q \arrow[dr, "\uplus"] & \\
h \models ([\![\pi]\!]Q) * R \arrow[ur, "\uplus"] \arrow[dr, "\uplus"] & & & h_Q \models Q * R \\
& h_2 \models R \arrow[r] & h_2 \models R \arrow[ur, "\uplus"] &
\end{tikzcd}$$

Together: $h \models [\![\pi]\!](Q * R)$

$h_Q = h[\mathit{val}(t)/\mathit{val}(u)]$ after executing Q

Soundness of Frame Rule

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Proof by structural induction over π.

Case $\pi_1 \,;\, \pi_2$

Assume:	$([\![\pi_1 \,;\, \pi_2]\!]Q) * R$
	$\big([\![\pi_1]\!]\big([\![\pi_2]\!]Q\big)\big) * R$
by ind. hyp.:	$[\![\pi_1]\!]\big(([\![\pi_2]\!]Q) * R\big)$
by ind. hyp.:	$[\![\pi_1]\!][\![\pi_2]\!](Q * R)$ using $\dfrac{\Box A \quad A \to B}{\Box B}$
	$[\![\pi_1 \,;\, \pi_2]\!](Q * R)$

Soundness of Frame Rule

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Proof by structural induction over π.

Remaining Cases: $x := [t]$, $\mathbf{?}\phi$, π^*

similar, left as exercise

Memory Allocation and Deallocation

Syntax: Two statements

$\textit{var} := \texttt{cons}(\textit{term}, \ldots, \textit{term})$ and $\texttt{dispose}(\textit{var})$

Memory Allocation and Deallocation

Karlsruhe Institute of Technology

Syntax: Two statements

var := `cons`(term, ..., term) and `dispose`(var)

Semantics: ρ and *fail*

$$((\beta, h), (\beta', h')) \in \rho(v := cons(t))$$

iff

$$\beta' = \beta[v/loc] \text{ and } h' = h \uplus \{(loc, val_\beta(t))\} \text{ and } loc \notin \text{dom } h$$

$$fail(v := cons(t_1, ...t_n)) = \emptyset$$

`cons` allocates n consecutive unused memory locations, stores the argument values there and returns the first memory location.

(See literature for general n-ary version)

Memory Allocation and Deallocation

Syntax: Two statements

var := `cons`(term, ..., term) and `dispose`(var)

Semantics: ρ and *fail*

$$((\beta, h), (\beta', h')) \in \rho(\texttt{dispose}(v))$$
$$\text{iff}$$
$$\beta' = \beta \text{ and } \beta(v) \in \operatorname{dom} h \text{ and } h' = h \setminus \{(\beta(v), h(\beta(v))\}$$

$$\mathit{fail}(\texttt{dispose}(v)) = \{(\beta, h) \mid \beta(v) \notin \operatorname{dom} h\}$$

`dispose` deallocates the allocated memory location v;
fails if an unallocated location is disposed.

Soundness of Frame Rule

$$\frac{([\![\pi]\!]Q) * R}{[\![\pi]\!](Q * R)} \text{ if } \mathit{Modifies}(\pi) \cap \mathit{Free}(R) = \emptyset$$

Proof by structural induction over π.

Case $x := \mathit{cons}(e)$

By assumption: For all $\mathit{loc} \notin \mathrm{dom}\, h_1$: Q holds after allocating loc in h_1.

Need to show: For all $\mathit{loc} \notin \mathrm{dom}\, h_1 \uplus h_2$: $Q * R$ holds after allocating loc in h.

This is a subset of the set in the assumption.

Decidability of Separation Logic

Decidable

Some restricted logics from Separation Logic are decidable.

1. Restricted arithmetic
2. No magic wand $-\!\!*$

They can be reduced to Monadic Second Order Logic over $\mathbb{N}$.
Equivalent to word emptiness of Büchi Automata.

The separating implication $-\!\!*$ makes undecidable.

Relatively complete

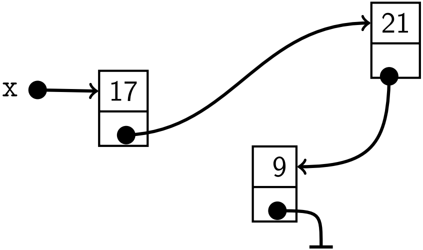
The calculus for Separation Logic is relatively complete.
Every correct program can be proved using an oracle for $\mathbb{N}$.

Application of Separation Logic

Abstraction Predicates

Use predicate symbols to abstract away from data structures

Example: Lists

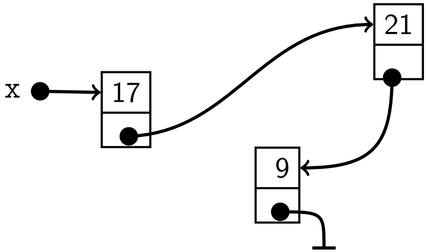


Abstraction Predicates

Use predicate symbols to abstract away from data structures

Example: Lists

$$\mathit{list}(x, \langle 17, 21, 9\rangle) \quad \leftrightarrow \quad (x \mapsto 17) * (x{+}1 \mapsto v) * (v \mapsto 21) * \ldots$$
$$\ldots * (v+1 \mapsto w) * (w \mapsto 9) * (w+1 \mapsto 0)$$



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General:

Recursive predicate *list*:

$$\forall x, v_1, \bar{v}.\ \mathit{list}(x, \langle v_1, \bar{v}\rangle) \leftrightarrow \exists n.\ ((x \mapsto v_1) * (x{+}1 \mapsto n) * \mathit{list}(n, \bar{v}))$$

Program Verification Using Separation Logic

- Verifast $\rightarrow$ Demo! (Bart Jacobs *et al.*, U Leuven)
  `https://www.cs.kuleuven.be/~bartj/verifast/`

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- jStar (M. Parkinson, now MS)
- SpaceInvader, YNot, HOLFoot, ..., ...

Discussion

Advantages of Separation Logic

+ Functional and frame specification combined – no extra consideration needed
+ Frame rule!
+ Abstraction Predicates are nice way of abstraction

Disadvantages of Separation Logic

– Functional and frame specification combined – no separation of concerns!
– All data must be hierarchically structured
– Complicated semantics of Sep Logic (c.f $-\!\!*$)