Formale Systeme II: Theorie

Dynamic Logic:
Propositional Dynamic Logic

SS 2022

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Slides partially by Prof. Dr. Peter H. Schmitt
Requirements for this topic

- Fundamental knowledge of discrete structures (graphs, (equivalence) relations)
- General understanding of syntax and semantics of propositional and first order Logic
- General understanding of semantical concepts like satisfiability, decidability of logics

for instance from lecture “Formale Systeme I’
Dynamic Logic(s)

Overview – a family of logics

Modal Logics

→ Propositional Dynamic Logic

→ Dynamic Logic

- Hybrid DL
- Java DL

Modal Logics: → Formal Systems I (recap here)
Java DL: Logic used in KeY
→ lecture “Formal Systems II – Applications”
Goals

We get to know **Dynamic Logic** as . . .

- abstract reasoning framework for descriptions of actions
Goals

We get to know Dynamic Logic as ... 

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- means to formalise and reason about semantics of programs
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- vehicle for examining/proving theoretical results on program reasoning
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- abstract reasoning framework for descriptions of actions
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  - what is decidable, what is not?
  - relative completeness
- concept of program verification on a while language
- logic for verification engines for realworld programming languages
- *Formale Systeme II*
  Vorlesungsskript
  Peter H. Schmitt
  → Website

- *Dynamic Logic*
  Series: Foundations of Computing
  David Harel, Dexter Kozen and Jerzy Tiuryn
  MIT Press
  → Department Library
Still an Active Field . . .

From the table of contents

- A Dynamic Logic for Learning Theory (Baltag et al.)
- Axiomatization and Computability of a Variant of Iteration-Free PDL with Fork (Balbiani et al.)
- Dynamic Preference Logic as a Logic of Belief Change (Souza et al.)
- Dynamic Logic: A Personal Perspective (*Vaughan Pratt*)
- . . .
Motivating Example
Introductory Example
Introductory Example

The Towers of Hanoi
Introductory Example

The Towers of Hanoi
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The Towers of Hanoi
The Instructions

1. Move alternatingly the smallest disk and another one.
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2. If moving the smallest disk put it on the stack it did not come from in its previous move.
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More formally:
sequence of actions

\[ \text{moveS} ; \text{moveO} ; \text{moveS} ; \text{moveO} ; \ldots \]
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more concisely:

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sequence of actions

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more concisely:

\[ (\text{moveS ; moveO})^* \]

improved:

\[ \text{moveS ; testForStop ; (moveO ; moveS ; testForStop)}^* \]
Properties

**Atomic statement**: $S_1$ true iff smallest piece on first stack
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**Moving away**

(1) $S_1 \rightarrow \langle \text{moveS} \rangle \neg S_1$

... after moving the smallest, it is no longer on the first stack
Properties

**Atomic statement:** $S_1$ true iff smallest piece on first stack

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**Moving other**

(2) $S_1 \rightarrow \langle moveO \rangle S_1$
... after moving something else, it is still on the first stack
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Conclusions from (1) and (2)

$S_1 \rightarrow \langle moveO \ ; \ moveS \rangle \neg S_1$
$S_1 \rightarrow \langle (moveO)^* \ ; \ moveS \rangle \neg S_1$
**Properties**

**Atomic statement:** $S_1$ true iff smallest piece on first stack

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**Conclusions from (1) and (2)**

$S_1 \rightarrow \langle moveO ; moveS \rangle \neg S_1$  
$S_1 \rightarrow \langle (moveO)^* ; moveS \rangle \neg S_1$

**THAT IS DYNAMIC LOGIC**
Recap: Modal Logic
Recap: Modal Logic

Syntax/semantics of dynamic logic build on top of modal logic.

Syntax:
Recap: Modal Logic

Syntax/semantics of dynamic logic build on top of modal logic.

Syntax:
- Signature $\Sigma$: set of propositional variables
Recap: Modal Logic

Syntax/semantics of dynamic logic build on top of modal logic.

**Syntax:**
- Signature $\Sigma$: set of propositional variables
- $\text{Fml}_\Sigma^{mod}$ smallest set with:
  - $\text{true}, \text{false} \in \text{Fml}_\Sigma^{mod}$
  - $A, B \in \text{Fml}_\Sigma^{mod} = \Rightarrow A \land B, A \lor B, A \rightarrow B \in \text{Fml}_\Sigma^{mod}$
  - $A \in \text{Fml}_\Sigma^{mod} = \Rightarrow \Box A, \Diamond A \in \text{Fml}_\Sigma^{mod}$
  - Pronounced “Box” and “Diamond”
Recap: Modal Logic

Syntax/semantics of dynamic logic build on top of modal logic.

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- pronounced “Box” and “Diamond”
Kripke Semantics

Modal logic formulas are interpreted in a system of multiple possible *worlds* and an *accessibility relation* between them.
Recap: Modal Logic – Semantics

**Kripke Semantics**
Modal logic formulas are interpreted in a system of multiple possible **worlds** and an **accessibility relation** between them.

**Kripke Frame** $(S, R)$:
- Set $S$ of **worlds** (or **states**)
- Relation $R \subseteq S \times S$, the **accessibility relation**
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Kripke Semantics

Modal logic formulas are interpreted in a system of multiple possible worlds and an accessibility relation between them.

Kripke Frame \((S, R)\):
- Set \(S\) of worlds (or states)
- Relation \(R \subseteq S \times S\), the accessibility relation

Kripke Structure \((S, R, I)\):
- Given a signature \(\Sigma\)
- Kripke Frame \((S, R)\)
- Interpretation \(I : S \rightarrow 2^\Sigma\)
Recap: Modal Logic – Semantics

For a signature $\Sigma$ and Kripke structure $(S, R, I)$

$I, s \models \varphi \iff$ Formula $\varphi$ holds in state $s \in S$

$I \models \varphi \iff$ Formula $\varphi$ holds in all states $s \in S$

$I, s \models p \iff p \in I(s)$ for $p \in \Sigma$
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$\models$ is as expected for $\land, \lor, \rightarrow, \neg$.

This means:  
$I, s \models \varphi \land \psi \iff I, s \models \varphi$ and $I, s \models \psi$
$I, s \models \varphi \lor \psi \iff I, s \models \varphi$ or $I, s \models \psi$
$I, s \models \varphi \rightarrow \psi \iff I, s \models \varphi$ implies $I, s \models \psi$
$I, s \models \neg \varphi \iff$ not $I, s \models \varphi$
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$I, s \models \Box \varphi \iff I, s' \models \varphi$ for all $s' \in S$ with $(s, s') \in R$
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- $I, s \models \Box \varphi \iff I, s' \models \varphi$ for all $s' \in S$ with $(s, s') \in R$
- $I, s \models \Diamond \varphi \iff I, s' \models \varphi$ for some $s' \in S$ with $(s, s') \in R$
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$I, s \models p$ $\iff$ $p \in I(s)$ for $p \in \Sigma$

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Applications of modal logics

Logics of *necessity* and *possibility* – philosophy.
Recap: Modal Logic – Semantics

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Meaning of Modalities:

Modal

□A It is necessary that . . .

♦A It is possible that . . .
Recap: Modal Logic – Semantics

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Logics of *necessity* and *possibility* – philosophy.

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**Epistemic** (logic of knowledge)

$\Box A$  I know that . . .

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Dynamic Logic
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- “Dynamic”: systematically changing evaluation context (by programs)
Dynamic Logic

- “Dynamic”: systematically changing evaluation context (by programs)
- “Programs” are composite actions

State change descriptions are explicit part of the logical language. There are two interdependent “sublanguages”:

1. Formulas
2. Programs

Extends modal logic
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  There are two interdependent “sublanguages”:
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  2. Programs
- Extends modal logic
More than one modality

Multi-modal logic

Have different Box operators with different accessibility relations:

\[ \Box_\alpha, \Box_\beta, \Box_\gamma, \ldots \]

(→ basic actions ins “Towers of Hanoi” )
More than one modality

Multi-modal logic

Have different Box operators with different accessibility relations:

$$\Box_\alpha, \Box_\beta, \Box_\gamma, \ldots$$

($\rightarrow$ basic actions ins “Towers of Hanoi”)

Propositional Dynamic Logic (PDL):

- Signature $\Sigma$ of propositional variables
- Set $A = \{\alpha, \beta, \ldots\}$ of atomic actions/programs
- We write $[\alpha]$ instead of $\Box_\alpha$
Compose Programs

Atomic programs can be composed into larger programs
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Atomic programs can be into composed into larger programs

For a given signature $\Sigma$ and atomic programs $A$, the set of programs $\Pi_{\Sigma,A}$ is the smallest set such that

1. $\mathcal{A} \subseteq \Pi_{\Sigma,A}$, atomic programs
2. $p, q \in \Pi_{\Sigma,A} \Rightarrow (p; q) \in \Pi_{\Sigma,A}$, sequential composition
3. $p, q \in \Pi_{\Sigma,A} \Rightarrow (p \cup q) \in \Pi_{\Sigma,A}$, nondeterministic choice
4. $p \in \Pi_{\Sigma,A} \Rightarrow p^* \in \Pi_{\Sigma,A}$, indeterminate iteration
5. $F \in \text{Fml}_{\text{PDL},A} \Rightarrow ?F \in \Pi_{\Sigma,A}$, tests

Regular Programs = Regular Expressions over atomic programs and tests
Compose Programs

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Compose Programs

Atomic programs can be composed into larger programs.

For a given signature $\Sigma$ and atomic programs $A$, the set of programs $\Pi_{\Sigma,A}$ is the smallest set such that:

1. $A \subseteq \Pi_{\Sigma,A}$  \hspace{1cm} \text{atomic programs}
2. $p, q \in \Pi_{\Sigma,A} \implies (p ; q) \in \Pi_{\Sigma,A}$  \hspace{1cm} \text{sequential composition}
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Composability of Programs

Atomic programs can be composed into larger programs.

For a given signature Σ and atomic programs A, the set of programs Π_{Σ,A} is the smallest set such that:

1. \( A \subseteq \Pi_{Σ,A} \)
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3. \( p, q \in \Pi_{Σ,A} \implies (p \cup q) \in \Pi_{Σ,A} \)
4. \( p \in \Pi_{Σ,A} \implies p^* \in \Pi_{Σ,A} \)

- \( \Pi_{Σ,A} \) is the set of programs.
- Atomic programs:
- Sequential composition
- Nondeterministic choice
- Indeterminant iteration
Compose Programs

Atomic programs can be composed into larger programs.

For a given signature $\Sigma$ and atomic programs $A$, the set of programs $\Pi_{\Sigma,A}$ is the smallest set such that:

1. $A \subseteq \Pi_{\Sigma,A}$
2. $p, q \in \Pi_{\Sigma,A} \implies (p ; q) \in \Pi_{\Sigma,A}$ (sequential composition)
3. $p, q \in \Pi_{\Sigma,A} \implies (p \cup q) \in \Pi_{\Sigma,A}$ (nondeterministic choice)
4. $p \in \Pi_{\Sigma,A} \implies p^* \in \Pi_{\Sigma,A}$ (indeterminate iteration)
5. $F \in \mathsf{Fml}_{\Sigma,A}^{PDL} \implies \mathsf{?F} \in \Pi_{\Sigma,A}$ (tests)
**Composing Programs**

Atomic programs can be composed into larger programs. For a given signature \( \Sigma \) and atomic programs \( A \), the set of programs \( \Pi_{\Sigma,A} \) is the smallest set such that:

1. \( A \subseteq \Pi_{\Sigma,A} \)  
   - **atomic programs**

2. \( p, q \in \Pi_{\Sigma,A} \implies (p ; q) \in \Pi_{\Sigma,A} \)  
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5. \( F \in \text{Fml}_{PDL}^{\Sigma,A} \implies ?F \in \Pi_{\Sigma,A} \)  
   - **tests**

**Regular Programs**

Regular Expressions over atomic programs and tests
PDL – Formulae

For a given signature $\Sigma$ and atomic programs $A$, the set of formulae $\textit{Fml}_{\Sigma,A}^{\textit{PDL}}$ is the smallest set such that

1. $\textit{true}, \textit{false} \in \textit{Fml}_{\Sigma,A}^{\textit{PDL}}$
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For a given signature $\Sigma$ and atomic programs $A$, the set of formulae $Fml_{\Sigma, A}^{PDL}$ is the smallest set such that

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For a given signature $\Sigma$ and atomic programs $A$, the set of formulae $Fml^{PDL}_{\Sigma,A}$ is the smallest set such that

1. $true, false \in Fml^{PDL}_{\Sigma,A}$
2. $\Sigma \subseteq Fml^{PDL}_{\Sigma,A}$
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4. $P \in \Pi_{\Sigma,A}, \varphi \in Fml^{PDL}_{\Sigma,A} \implies [P]\varphi, \langle P \rangle \varphi \in Fml^{PDL}_{\Sigma,A}$
PDL – Formulae

For a given signature $\Sigma$ and atomic programs $A$, the set of formulae $Fml_{\Sigma,A}^{PDL}$ is the smallest set such that

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4. $P \in \Pi_{\Sigma,A}, \varphi \in Fml_{\Sigma,A}^{PDL} \implies [P]\varphi, \langle P \rangle \varphi \in Fml_{\Sigma,A}^{PDL}$

Programs and Formulae are mutually dependent definitions and must be seen simultaneously.
PDL Formulas – Examples

→ Towers of Hanoi

\[ A = \{ \text{moveS, moveO} \}, \quad \Sigma = \{ S1 \} \]

\[ S1 \rightarrow ((\text{moveO})^* ; \text{moveS}) \neg S1 \]
→ Towers of Hanoi

\[ A = \{ \text{moveS, moveO} \}, \quad \Sigma = \{ S1 \} \]

\[ S1 \rightarrow \langle (\text{moveO})^* ; \text{moveS} \rangle \neg S1 \]

multi-level and nested modalities

\[ A = \{ \alpha, \beta \}, \quad \Sigma = \{ P, Q \} \]

\[
\begin{align*}
[\alpha \cup (\text{?}P ; \beta)^*]Q \\
[\alpha]P \rightarrow [\alpha^*]P \\
[\alpha]\langle \beta \rangle (P \rightarrow [\alpha^*]Q) \\
[\alpha ; \text{?}\langle \beta \rangle P ; \beta]Q
\end{align*}
\]
PDL – Semantics

Given a signature $\Sigma$ and atomic programs $A$

(multi-modal propositional) Kripke frame $(S, \rho)$

- set of states $S$
- function $\rho : A \rightarrow 2^{S \times S}$ accessibility relations for atomic programs
PDL – Semantics

Given a signature $\Sigma$ and atomic programs $A$

**(multi-modal propositional) Kripke frame** $(S, \rho)$
- set of states $S$
- function $\rho : A \rightarrow 2^{S \times S}$ accessibility relations for atomic programs

**Kripke structure** $(S, \rho, I)$
- Kripke frame $(S, \rho)$
- interpretation $I : S \rightarrow 2^\Sigma$

$\Rightarrow$ same as for modal logic
Extension of $\rho$

from $\rho : A \rightarrow 2^{S^2}$ to $\rho : \Pi_{\Sigma,A} \rightarrow 2^{S^2}$
**Extension of $\rho$**

from $\rho : A \rightarrow 2^{S^2}$ to $\rho : \Pi_{\Sigma,A} \rightarrow 2^{S^2}$

- $\rho(\alpha)$ base case for $\alpha \in A$
- $\rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2)$
## Extension of $\rho$

from $\rho : A \rightarrow 2^{S^2}$ to $\rho : \Pi_{\Sigma,A} \rightarrow 2^{S^2}$

<table>
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<tr>
<th>Expression</th>
<th>Description</th>
</tr>
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<tr>
<td>$\rho(\alpha)$</td>
<td>base case for $\alpha \in A$</td>
</tr>
<tr>
<td>$\rho(\pi_1 \cup \pi_2)$</td>
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</tr>
<tr>
<td>$\rho(\pi_1 ; \pi_2)$</td>
<td>$= \rho(\pi_1) ; \rho(\pi_2)$</td>
</tr>
<tr>
<td></td>
<td>$= {(s, s') \mid \text{ex. } t \text{ with } (s, t) \in \rho(\pi_1) \text{ and } (t, s') \in \rho(\pi_2)}$</td>
</tr>
</tbody>
</table>
PDL – Program Semantics

Extension of \( \rho \)

from \( \rho : A \to 2^{S^2} \) to \( \rho : \Pi_{\Sigma, A} \to 2^{S^2} \)

\[
\begin{align*}
\rho(\alpha) & \quad \text{base case for } \alpha \in A \\
\rho(\pi_1 \cup \pi_2) & = \rho(\pi_1) \cup \rho(\pi_2) \\
\rho(\pi_1 ; \pi_2) & = \rho(\pi_1) \cdot \rho(\pi_2) \\
& = \{(s, s') \mid \text{ex. } t \text{ with } (s, t) \in \rho(\pi_1) \text{ and } (t, s') \in \rho(\pi_2)\} \\
\rho(\pi^*) & = \text{rtcl}(\rho(\pi)) = \bigcup_{n=0}^{\infty} \rho(\pi)^n \quad \text{refl. transitive closure} \\
& = \{(s_0, s_n) \mid \text{ex. } n \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ for } 0 \leq i < n\}
\end{align*}
\]
Extension of $\rho$

from $\rho : A \rightarrow 2^{S^2}$ to $\rho : \Pi_{\Sigma,A} \rightarrow 2^{S^2}$

$$\rho(\alpha) \quad \text{base case for } \alpha \in A$$

$$\rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2)$$

$$\rho(\pi_1 ; \pi_2) = \rho(\pi_1) ; \rho(\pi_2)$$

$$= \{(s, s') \mid \text{ex. } t \text{ with } (s, t) \in \rho(\pi_1) \text{ and } (t, s') \in \rho(\pi_2)\}$$

$$\rho(\pi^*) = \text{rtcl}(\rho(\pi)) = \bigcup_{n=0}^{\infty} \rho(\pi)^n \quad \text{refl. transitive closure}$$

$$= \{(s_0, s_n) \mid \text{ex. } n \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ for } 0 \leq i < n\}$$

$$\rho(?F) = \{(s, s) \mid I, s \models F\}$$
For a signature $\Sigma$, basic programs $A$ and Kripke structure $(S, \rho, I)$
For a signature $\Sigma$, basic programs $A$ and Kripke structure $(S, \rho, I)$

$$I, s \models p \iff p \in I(s) \quad \text{for } p \in \Sigma$$
PDL – Semantics

For a signature $\Sigma$, basic programs $A$ and Kripke structure $(S, \rho, I)$

\[
l, s \models p \iff p \in I(s) \quad \text{for } p \in \Sigma
\]

$\models$ is as expected for $\land, \lor, \rightarrow, \neg$. 
For a signature $\Sigma$, basic programs $A$ and Kripke structure $(S, \rho, I)$

$l, s \vDash p \iff p \in l(s)$ for $p \in \Sigma$

$\vDash$ is as expected for $\land, \lor, \rightarrow, \neg$.

$l, s \vDash [\pi] \varphi \iff l, s' \vDash \varphi$ for all $s' \in S$ with $(s, s') \in \rho(\pi)$
PDL – Semantics

For a signature $\Sigma$, basic programs $A$ and Kripke structure $(S, \rho, I)$

$I, s \models p \iff p \in I(s)$ for $p \in \Sigma$

$\models$ is as expected for $\land, \lor, \rightarrow, \neg$.

$I, s \models [\pi] \varphi \iff I, s' \models \varphi$ for all $s' \in S$ with $(s, s') \in \rho(\pi)$

$I, s \models \langle \pi \rangle \varphi \iff I, s' \models \varphi$ for some $s' \in S$ with $(s, s') \in \rho(\pi)$
Tautologies

Dual operators

\[ [\pi] \varphi \leftrightarrow \neg \langle \pi \rangle \neg \varphi \]
### Tautologies

#### Dual operators

<table>
<thead>
<tr>
<th>[π]φ</th>
<th>⇔</th>
<th>¬⟨π⟩¬φ</th>
</tr>
</thead>
</table>

| [π₁ ∪ π₂]φ | ⇔ | [π₁]φ ∧ [π₂]φ |
| ⟨?ψ⟩φ | ⇔ | ψ → φ |
| ⟨π∗⟩φ | ⇔ | φ ∨ ⟨π;π∗⟩φ |

All tautologies for modal logic $\mathcal{K}$

Beckert, Ulbrich – Formale Systeme II: Theorie 25/61
Tautologies

Dual operators

$$[\pi] \phi \iff \neg \langle \pi \rangle \neg \phi$$

- $$[\pi_1 ; \pi_2] \phi \iff [\pi_1][\pi_2] \phi$$
Tautologies

Dual operators

\[ [\pi] \varphi \iff \neg \langle \pi \rangle \neg \varphi \]

- \[ [\pi_1 ; \pi_2] \varphi \iff [\pi_1][\pi_2] \varphi \]
- \[ [\pi_1 \cup \pi_2] \varphi \iff [\pi_1] \varphi \land [\pi_2] \varphi \]
Tautologies

Dual operators

\([\pi] \varphi \iff \neg \langle \pi \rangle \neg \varphi\)

- \([\pi_1 ; \pi_2] \varphi \iff [\pi_1][\pi_2] \varphi\)
- \([\pi_1 \cup \pi_2] \varphi \iff [\pi_1] \varphi \land [\pi_2] \varphi\)
- \([?\psi] \varphi \iff \psi \rightarrow \varphi\)
Tautologies

Dual operators

\[[\pi]\varphi \leftrightarrow \neg(\pi)\neg\varphi\]

- \[[\pi_1 ; \pi_2]\varphi \leftrightarrow [\pi_1][\pi_2]\varphi\]
- \[[\pi_1 \cup \pi_2]\varphi \leftrightarrow [\pi_1]\varphi \land [\pi_2]\varphi\]
- \[[?\psi]\varphi \leftrightarrow \psi \rightarrow \varphi\]
- \[[\pi^*]\varphi \leftrightarrow \varphi \land [\pi ; \pi^*]\varphi\]
Tautologies

Dual operators

\[ [\pi] \varphi \iff \neg \langle \pi \rangle \neg \varphi \]

- \([\pi_1 ; \pi_2] \varphi \iff [\pi_1][\pi_2] \varphi\]
- \([\pi_1 \cup \pi_2] \varphi \iff [\pi_1] \varphi \land [\pi_2] \varphi\]
- \([?\psi] \varphi \iff \psi \to \varphi\]
- \([\pi^*] \varphi \iff \varphi \land [\pi ; \pi^*] \varphi\]
- \(\langle \pi_1 ; \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi\]
Tautologies

Dual operators

\[[\pi]\varphi \iff \neg \langle \pi \rangle \neg \varphi\]

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- \[[?\psi]\varphi \iff \psi \to \varphi\]
- \[[\pi^*]\varphi \iff \varphi \land [\pi ; \pi^*]\varphi\]

- \langle \pi_1 ; \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi\]
- \langle \pi_1 \cup \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \varphi \lor \langle \pi_2 \rangle \varphi\]
### Tautologies

#### Dual operators

\[
[\pi]\varphi \iff \neg\langle\pi\rangle\neg\varphi
\]

- \([\pi_1 ; \pi_2]\varphi \iff [\pi_1][\pi_2]\varphi\)
- \([\pi_1 \cup \pi_2]\varphi \iff [\pi_1]\varphi \land [\pi_2]\varphi\)
- \([?\psi]\varphi \iff \psi \rightarrow \varphi\)
- \([\pi^*]\varphi \iff \varphi \land [\pi ; \pi^*]\varphi\)

- \(\langle\pi_1 ; \pi_2\rangle\varphi \iff \langle\pi_1\rangle\langle\pi_2\rangle\varphi\)
- \(\langle\pi_1 \cup \pi_2\rangle\varphi \iff \langle\pi_1\rangle\varphi \lor \langle\pi_2\rangle\varphi\)
- \(\langle?\psi\rangle\varphi \iff \psi \land \varphi\)
Tautologies

Dual operators

\[ [\pi] \varphi \iff \neg \langle \pi \rangle \neg \varphi \]

- \[ [\pi_1 ; \pi_2] \varphi \iff [\pi_1][\pi_2] \varphi \]
- \[ [\pi_1 \cup \pi_2] \varphi \iff [\pi_1] \varphi \land [\pi_2] \varphi \]
- \[ [\psi] \varphi \iff \psi \rightarrow \varphi \]
- \[ [\pi^*] \varphi \iff \varphi \land [\pi ; \pi^*] \varphi \]
- \[ \langle \pi_1 ; \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi \]
- \[ \langle \pi_1 \cup \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \varphi \lor \langle \pi_2 \rangle \varphi \]
- \[ \langle \psi \rangle \varphi \iff \psi \land \varphi \]
- \[ \langle \pi^* \rangle \varphi \iff \varphi \lor \langle \pi ; \pi^* \rangle \varphi \]
Tautologies

Dual operators

\([\pi] \varphi \iff \neg \langle \pi \rangle \neg \varphi\)

- \([\pi_1 ; \pi_2] \varphi \iff [\pi_1][\pi_2] \varphi\)
- \([\pi_1 \cup \pi_2] \varphi \iff [\pi_1] \varphi \wedge [\pi_2] \varphi\)
- \([?\psi] \varphi \iff \psi \to \varphi\)
- \([\pi^*] \varphi \iff \varphi \wedge [\pi ; \pi^*] \varphi\)

- \(\langle \pi_1 ; \pi_2 \rangle \varphi \iff \langle \pi_1 \rangle \langle \pi_2 \rangle \varphi\)
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- \(\langle ?\psi \rangle \varphi \iff \psi \land \varphi\)
- \(\langle \pi^* \rangle \varphi \iff \varphi \lor \langle \pi ; \pi^* \rangle \varphi\)

- all tautologies for modal logic \(K\)
A Calculus for Propositional Dynamic Logic

Axioms

All propositional tautologies

\[ [\pi](\varphi \rightarrow \psi) \rightarrow ([\pi]\varphi \rightarrow [\pi]\psi) \]  
\[ ([\pi]\varphi \wedge [\pi]\psi) \leftrightarrow [\pi_1][\pi_2]\varphi \]  
\[ [\pi_1; \pi_2]\varphi \leftrightarrow [\pi_1][\pi_2]\varphi \]  
\[ [\pi_1 \cup \pi_2]\varphi \leftrightarrow [\pi_1]\varphi \wedge [\pi_2]\varphi \]  
\[ [\pi^*]\varphi \leftrightarrow \varphi \wedge [\pi][\pi^*]\varphi \]  
\[ \varphi \wedge [\pi^*](\varphi \rightarrow [\pi]\varphi) \rightarrow [\pi^*]\varphi \]

Rules

\[ \varphi, \varphi \rightarrow \psi \]
\[ \frac{\varphi}{[\pi]\varphi} \]  
\[ \frac{\varphi}{\psi} \]  

(MP)  
(GEN)
The presented calculus is sound and complete.
Theorem

The presented calculus is sound and complete.

Proof

or
Theorem

The presented calculus is sound and complete.

Proof

See e.g., pp. 559-560 in David Harel’s article *Dynamic Logic* in the *Handbook of Philosophical Logic, Volume II*, published by D. Reidel in 1984.
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See e.g., pp. 559-560
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or

D. Harel, D. Kozen and J. Tiuryn
*Dynamic Logic*
in *Handbook of Philosophical Logic, 2nd edition, volume 4*
Higher level program constructors

Syntactic Sugar

- PDL syntax has elementary program operators
- Enrich it by defining new operators ("macros")
Higher level program constructors

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\[
\text{skip} := \ ?\text{true}
\]
Higher level program constructors

**Syntactic Sugar**

- PDL syntax has elementary program operators
- Enrich it by defining new operators ("macros")

```plaintext
skip := ?true
fail := ?false
```
Higher level program constructors

### Syntactic Sugar

- PDL syntax has elementary program operators
- Enrich it by defining new operators ("macros")

\[
\text{skip} := \ ?true \\
\text{fail} := \ ?false \\
\text{if } \varphi \text{ then } \alpha \text{ else } \beta := (\ ?\varphi ; \alpha ) \cup (\neg \varphi ; \beta )
\]
Higher level program constructors

Syntactic Sugar

- PDL syntax has elementary program operators
- Enrich it by defining new operators ("macros")

\[
\begin{align*}
\text{skip} & := \ ?true \\
\text{fail} & := \ ?false \\
\text{if } \varphi \text{ then } \alpha \text{ else } \beta & := (?\varphi ; \alpha) \cup (?\neg \varphi ; \beta) \\
\text{while } \varphi \text{ do } \alpha & := (?\varphi ; \alpha)^* ; ?\neg \varphi
\end{align*}
\]
More PDL Tautologies

\[\text{[skip]} \varphi \iff \varphi\]
More PDL Tautologies

\[ [\text{skip}] \varphi \iff \varphi \]

\[ \langle \text{skip} \rangle \varphi \iff \varphi \]
More PDL Tautologies

\[ [\text{skip}] \varphi \iff \varphi \]

\[ \langle \text{skip} \rangle \varphi \iff \varphi \]

\[ [\text{fail}] \varphi \iff \text{true} \]
More PDL Tautologies

\[ [\text{skip}] \varphi \iff \varphi \]
\[ \langle \text{skip} \rangle \varphi \iff \varphi \]
\[ [\text{fail}] \varphi \iff \text{true} \]
\[ \langle \text{fail} \rangle \varphi \iff \text{false} \]
More PDL Tautologies

\[
\begin{align*}
\text{[skip]} \varphi & \leftrightarrow \varphi \\
\langle \text{skip} \rangle \varphi & \leftrightarrow \varphi \\
\text{[fail]} \varphi & \leftrightarrow \text{true} \\
\langle \text{fail} \rangle \varphi & \leftrightarrow \text{false} \\
\text{[if } \varphi \text{ then } \alpha \text{ else } \beta \text{]} \psi & \leftrightarrow (\varphi \rightarrow [\alpha] \psi) \land (\neg \varphi \rightarrow [\beta] \psi)
\end{align*}
\]
More PDL Tautologies

\[
\begin{align*}
[\text{skip}] \varphi & \iff \varphi \\
⟨\text{skip}⟩ \varphi & \iff \varphi \\
[\text{fail}] \varphi & \iff \text{true} \\
⟨\text{fail}⟩ \varphi & \iff \text{false} \\
[\text{if } \varphi \text{ then } \alpha \text{ else } \beta] \psi & \iff (\varphi \rightarrow [\alpha] \psi) \land (\neg \varphi \rightarrow [\beta] \psi) \\
⟨\text{if } \varphi \text{ then } \alpha \text{ else } \beta⟩ \psi & \iff (\varphi \rightarrow ⟨\alpha⟩ \psi) \land (\neg \varphi \rightarrow ⟨\beta⟩ \psi)
\end{align*}
\]
More PDL Tautologies

\[ [\text{skip}] \varphi \iff \varphi \]
\[ \langle \text{skip} \rangle \varphi \iff \varphi \]
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\[ \langle \text{fail} \rangle \varphi \iff \text{false} \]
\[ [\text{if } \varphi \text{ then } \alpha \text{ else } \beta] \psi \iff (\varphi \rightarrow [\alpha] \psi) \land (\neg \varphi \rightarrow [\beta] \psi) \]
\[ \langle \text{if } \varphi \text{ then } \alpha \text{ else } \beta \rangle \psi \iff (\varphi \rightarrow \langle \alpha \rangle \psi) \land (\neg \varphi \rightarrow \langle \beta \rangle \psi) \]
Decidability
Decidability

Is PDL decidable?

\[ \iff \]

Is there an algorithm that terminates on every input and computes whether a PDL-formula \( \phi \in Fml_{\Sigma,A}^{PDL} \) is satisfiable.

Answer: \text{YES}, PDL is decidable!
Decidability

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Answer:

YES, PDL is decidable!
General Idea:

\( \varphi \in Fml^{PDL} \) has a model \( \iff \) \( \varphi \) has a model of bounded size.

For every Kripke structure, a bounded Kripke structure can be defined which is indistinguishable for \( \varphi \).
Fischer and Ladner (1979)

General Idea:
\( \varphi \in Fml^{PDL} \) has a model \( \iff \varphi \) has a model of bounded size.

For every Kripke structure, a bounded Kripke structure can be defined which is indistinguishable for \( \varphi \).

Preliminary lemma: Decidability for modal logic
The proof idea is the same, yet simpler.
Fischer-Ladner Closure

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<th>Reduced syntax</th>
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<td>Only connectors $\rightarrow$, $false$, $\Box$ are allowed $\Rightarrow$ simplifies proofs.</td>
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Fischer-Ladner Closure

Reduced syntax
Only connectors $\rightarrow$, $false$, $\square$ are allowed $\Rightarrow$ simplifies proofs.

Operator

$$FL^{mod} : Fml^{mod} \rightarrow 2^{Fml^{mod}}$$

assigns to $\varphi$ the set of subformulas of $\varphi$. 
Fischer-Ladner Closure

**Reduced syntax**
Only connectors $\rightarrow$, \textit{false}, $\Box$ are allowed $\Rightarrow$ simplifies proofs.

**Operator**

\[ FL^{\text{mod}} : Fml^{\text{mod}} \rightarrow 2^{Fml^{\text{mod}}} \]

assigns to $\varphi$ the set of subformulas of $\varphi$.

\[
FL^{\text{mod}}(\varphi \rightarrow \psi) = \{\varphi \rightarrow \psi\} \cup FL^{\text{mod}}(\varphi) \cup FL^{\text{mod}}(\psi)
\]

\[
FL^{\text{mod}}(\text{false}) = \{\text{false}\}
\]

\[
FL^{\text{mod}}(p) = \{p\} \quad p \in \Sigma
\]

\[
FL^{\text{mod}}(\Box \varphi) = \{\Box \varphi\} \cup FL^{\text{mod}}(\varphi)
\]
Fischer-Ladner Closure

Reduced syntax

Only connectors $\rightarrow$, $\textit{false}$, $\square$ are allowed $\Rightarrow$ simplifies proofs.

Operator

$$FL^{mod} : Fml^{mod} \rightarrow 2^{Fml^{mod}}$$

assigns to $\varphi$ the set of subformulas of $\varphi$.

$$FL^{mod}(\varphi \rightarrow \psi) = \{\varphi \rightarrow \psi\} \cup FL^{mod}(\varphi) \cup FL^{mod}(\psi)$$

$$FL^{mod}(\textit{false}) = \{\textit{false}\}$$

$$FL^{mod}(p) = \{p\} \quad p \in \Sigma$$

$$FL^{mod}(\square \varphi) = \{\square \varphi\} \cup FL^{mod}(\varphi)$$

Observation

$$|FL^{mod}(\varphi)| \leq |\varphi|$$
For a Kripke structure \( S, R, I \) define a bounded structure \( \tilde{S}, \tilde{R}, \tilde{I} \) with

\[
S, R, I, s \models \varphi \iff \tilde{S}, \tilde{R}, \tilde{I}, \tilde{s} \models \varphi
\]
Filtration for modal logic

Filtration

For a Kripke structure $S, R, I$ define a bounded structure $\tilde{S}, \tilde{R}, \tilde{I}$ with

$$S, R, I, s \models \varphi \iff \tilde{S}, \tilde{R}, \tilde{I}, \tilde{s} \models \varphi$$

Central Idea

States are undistinguishable for $\varphi$ if they are equal on $FL^{mod}(\varphi)$. 
Filtration for modal logic

**Filtration**

For a Kripke structure $S, R, I$ define a bounded structure $\tilde{S}, \tilde{R}, \tilde{I}$ with

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**Central Idea**

States are **undistinguishable** for $\varphi$ if they are equal on $FL^{\text{mod}}(\varphi)$.

$$s \equiv t \iff (I, s \models \psi \iff I, t \models \psi \text{ for all } \psi \in FL^{\text{mod}}(\varphi))$$
Filtration for modal logic

Filtration

For a Kripke structure $S, R, I$ define a bounded structure $\tilde{S}, \tilde{R}, \tilde{I}$ with

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Central Idea

States are **undistinguishable** for $\varphi$ if they are equal on $FL^{mod}(\varphi)$.

$$s \equiv t \iff (I, s \models \psi \iff I, t \models \psi \text{ for all } \psi \in FL^{mod}(\varphi))$$

$$\tilde{s} := \{s' \mid s' \equiv s\} \quad \ldots \quad \text{equivalence classes}$$

$$\tilde{S} := \{\tilde{s} \mid s \in S\}$$

$$\tilde{R} := \{(\tilde{s}, \tilde{s'}) \mid (s, s') \in R\}$$

$$\tilde{I}(\tilde{s}) := I(s)$$
Fischer-Ladner Filtration

\[ \tilde{s} := \{ s' \mid s' \equiv s \} \]
\[ \tilde{S} := \{ \tilde{s} \mid s \in S \} \]
\[ \tilde{R} := \{ (\tilde{s}, \tilde{t}) \mid (s, t) \in R \} \]
\[ \tilde{l}(s) := l(s) \]

Lemma (proved by structural induction)

\[ |\tilde{S}| \leq 2|FL| \mod (\varphi) \leq 2|\varphi| \]

Theorem (small model property)

For any PDL formula \( \varphi \) it can be decided if \( \varphi \) is satisfiable by inspecting a finite number (those up to size \( 2|\varphi| \)) of models.
Fischer-Ladner Filtration

\[ \tilde{s} := \{ s' \mid s' \equiv s \} \]
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\[ \tilde{I}(\tilde{s}) := I(s) \]

Lemma

\[ |\tilde{S}| \leq 2^{|FL^{\text{mod}}(\varphi)|} \leq 2^{|\varphi|} \]
Fischer-Ladner Filtration

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\[ \tilde{S} := \{ \tilde{s} \mid s \in S \} \]
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**Lemma**

\[ |\tilde{S}| \leq 2^{|FL^\text{mod}(\phi)|} \leq 2^{|\phi|} \]

**Lemma (proved by structural induction)**

\[ S, R, I, s \models \phi \iff \tilde{S}, \tilde{R}, \tilde{I}, \tilde{s} \models \phi \]
Fischer-Ladner Filtration

\[ \tilde{s} := \{s' \mid s' \equiv s\} \]

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Fischer-Ladner Closure for PDL

**Operator**

\[ FL : \mathcal{Fml}^{PDL} \rightarrow 2^{\mathcal{Fml}^{PDL}} \]

\( FL(\varphi) \) smallest set satisfying

1. \( \varphi \in FL(\varphi) \)
2. \( (\psi_1 \rightarrow \psi_2) \in FL(\varphi) \) \( \Rightarrow \) \( \psi_1 \in FL(\varphi) \) and \( \psi_2 \in FL(\varphi) \)
3. \( [\pi]\psi \in FL(\varphi) \) \( \Rightarrow \) \( \psi \in FL(\varphi) \)
4. \( [\pi_1; \pi_2]\psi \in FL(\varphi) \) \( \Rightarrow \) \( [\pi_1][\pi_2]\psi \in FL(\varphi) \)
5. \( [\pi_1 \cup \pi_2]\psi \in FL(\varphi) \) \( \Rightarrow \) \( [\pi_1]\psi \in FL(\varphi) \) and \( [\pi_2]\psi \in FL(\varphi) \)
6. \( [\pi^*]\psi \in FL(\varphi) \) \( \Rightarrow \) \( [\pi][\pi^*]\psi \in FL(\varphi) \)
7. \( [?\psi_1]\psi_2 \in FL(\varphi) \) \( \Rightarrow \) \( \psi_1 \in FL(\varphi) \)

**Lemma (not obvious)**

\[ |FL(\varphi)| \leq |\varphi| \]
Fischer-Ladner Filtration

Same construction as for modal logic

extended:

\[ \tilde{\rho}(a) := \{(\tilde{s}, \tilde{t}) \mid (s, t) \in \rho(a)\} \quad \text{for all } a \in A \]
Fischer-Ladner Filtration

Same construction as for modal logic

extended: \( \tilde{\rho}(a) := \{ (\tilde{s}, \tilde{t}) \mid (s, t) \in \rho(a) \} \) for all \( a \in A \)

Lemma

\[ S, R, I, s \models \varphi \iff \tilde{S}, \tilde{R}, \tilde{I}, \tilde{s} \models \varphi \]
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Prove by structural induction: \(\leadsto\) lec. notes or [Harel et al., 6.4]

**A.** If \(\psi \in FL(\varphi)\) then \(s \models \psi\) iff \(\tilde{s} \models \psi\)
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Prove by structural induction: \(\leadsto\) lec. notes or [Harel et al., 6.4]

A. If $\psi \in FL(\varphi)$ then $s \models \psi$ iff $\tilde{s} \models \psi$

B1. $(s, t) \in \rho(\pi)$ implies $\tilde{(s, t)} \in \tilde{\rho}(\pi)$ for $[\pi]\psi \in FL(\varphi)$
Fischer-Ladner Filtration

Same construction as for modal logic

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Corollary

PDL has the small model property:
If \( \varphi \in Fml^{PDL} \) is satisfiable, it has a model with at most \( 2^{\|\varphi\|} \) states.
### Complexity

**Naive approach used for proof**

- $FL(\varphi) \in O(|\varphi|)$
Complexity

<table>
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<th>Naive approach used for proof</th>
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Naive approach used for proof

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- $|\tilde{S}| \leq 2^{FL(\varphi)} \in O(2^{|\varphi|})$ many states in filtration
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$\Rightarrow$ double exponential complexity
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One can do better:

Complexity of Deciding PDL

The decision problem for PDL is in EXPTIME:
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$\Rightarrow$ double exponential complexity

One can do better:

Complexity of Deciding PDL

The decision problem for PDL is in EXPTIME: can be decided by a deterministic algorithm in $O(2^{p(n)})$ for some polynomial $p$.

$\sim$[Harel et al. Ch. 8]
Deduction Theorem and Compactness
Logical Consequence

\[ M \subseteq \text{Fml}^{PDL}, \quad \varphi \in \text{Fml}^{PDL} \]

Global Consequence

\[ M \models^G \varphi : \iff \]
for all Kripke structures \((S, \rho, I)\):
\[ I, s \models M \text{ for all } s \in S \quad \text{implies} \quad I, s \models \varphi \text{ for all } s \in S \]

Local Consequence

\[ M \models^L \varphi : \iff \]
for all Kripke structures \((S, \rho, I)\):
\[ \text{for all } s \in S: \quad I, s \models M \text{ implies } I, s \models \varphi \]

Local consequence is stronger:

\[ M \models^L \varphi \quad \iff \quad M \models^G \varphi \]

Recall: In propositional logic:

\[ M \cup \{ \varphi \} \models \psi \iff M \models \varphi \rightarrow \psi \]
Deduction Theorem

**Recall:** In propositional logic:

\[
M \cup \{\varphi\} \models \psi \iff M \models \varphi \to \psi
\]

**Not valid for PDL:**

\[
p \models^G [\alpha]p \text{ but } \not\models^G p \to [\alpha]p
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1. Is \( \psi \models^G \varphi \) decidable for PDL?
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Lemma

\[ \psi \models^G \varphi \iff \models ([(\beta_1 \cup \ldots \cup \beta_k)^*] \psi) \rightarrow \varphi \]

with \( B := \{\beta_1, \ldots, \beta_k\} \) the atomic programs occurring in \( \psi, \varphi \).
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\( \iff \) simple \( \rightsquigarrow \) Exercise
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simple ⇔ Exercise

---

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Decidable:

The consequence problem \( \psi \models^G \varphi \) is decidable for PDL.
Compactness of PDL

Recall: Compactness Theorem

\[ M \models^G \varphi \iff \exists \text{ finite } E \subseteq M \text{ with } E \models^G \varphi \]

Holds for:
Propositional Logic, First Order Logic, **not** for higher order logic

PDL is not compact because it has transitive closure "built in."
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Counterexample for PDL

\[ M := \{ p \rightarrow [\underbrace{\alpha ; \ldots ; \alpha}_n]q \mid n \in \mathbb{N} \}, \quad \varphi := p \rightarrow [\alpha^*]q \]

\[ \text{n times} \]
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- \( E \subset M, \ E \models^G \varphi \) \quad ? \quad no
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Beckert, Ulbrich – Formale Systeme II: Theorie
Deducibility Problem in PDL

Quote:

[T]he problem of whether an arbitrary PDL formula $p$ is deducible from a single fixed axiom scheme is of extremely high degree of undecidability, namely $\Pi^1_1$-complete.

Meyer, Streett, Mirkowska:  
The Deducibility Problem in Propositional Dynamic Logic, 1981
Variants and Conclusion
Variant: Converse Programs

Idea: Add actions reverting action effects

Add further program constructor \( \cdot^{-1} \):

\[
\pi \in \Pi \implies \pi^{-1} \in \Pi
\]

with \( \rho(\pi^{-1}) = \rho(\pi)^{-1} \)
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Axiom schemes: for all \( \phi \in Fml^{PDL} \), \( \pi \in \Pi \)

- \( \phi \to [\pi] \langle \pi^{-1} \rangle \phi \)
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Complete

Adding the axioms to the known PDL calculus gives a correct and complete calculus for PDL with Converse.
Variant: Context-free Programs

Idea: Go beyond regular programs

Instead of regular programs, allow context-free grammar

Produced context-free grammar

\[ X ::= \alpha X \gamma \mid \beta \]

with

\[ L(X) = \{ \alpha^n \beta \gamma^n \mid n \in \mathbb{N} \} \]

Undecidability result

Validity is undecidable if instead of regular programs, context-free programs are allowed.

Expressiveness

Without fixed semantics of \( \mathbb{N} \), recursion is strictly more expressive than looping.
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For example:
Produced context-free grammar $X ::= \alpha X \gamma | \beta$
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Without fixed semantics of $\mathbb{N}$, recursion is strictly more expressive than looping.
A propositional Kripke structure $\mathcal{K} = (S, \rho, I)$ is determined by:

- $S$ \quad \text{the set of states}
- $\rho : A \rightarrow S \times S$ \quad \text{the accessibility relations for atomic programs}
- $I : S \rightarrow 2^\Sigma$ \quad \text{evaluation of propositional atoms in states}

Choose now: $S \subseteq 2^\Sigma$ the set of states

Strictly larger set of tautologies. Obviously decidable.

Evaluation of propositional variables fixes the state (and the accessibility of successor states)
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Then

\[
\langle \pi_{all} \rangle (state_U \land F) \rightarrow [\pi_{all}](state_U \rightarrow F)
\]

is true in all state vector Kripke structures.
Theorem

Let $H$ be the set of all formulas

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with the notation from the previous slide.
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Let $H$ be the set of all formulas

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1. $\{F\} \cup H$ is satisfiable iff $F$ is state vector satisfiable.
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1. $\{F\} \cup H$ is satisfiable iff $F$ is state vector satisfiable.
2. $H \models F$ iff $\models_{sv} F$. 
Propositional Dynamic Logic – Summary

- extension of modal logic
- abstract notion of actions / atomic logic statements
- regular programs, with non-deterministic choice and Kleene-interation
- correct and complete calculus for tautologies
- satisfiability is decidable (in EXPTIME)
- logic is not compact
- deducibility is utterly undecidable
- deduction theorem can be rescued
Detection of dynamic execution errors in IBM system automation’s rule-based expert system

An Application of PDL
[SinzEtAl02]

Carsten Sinz, Thomas Lumpp, Jürgen Schneider, and Wolfgang Küchlin:
Detection of dynamic execution errors in IBM System Automation’s rule-based expert system.
Context
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- z = zero downtime
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- $z = \text{zero downtime}$
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**System Automation**

Example: Flight booking center: 100s of users, many parallel apps
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- full automation of a data center
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- ...
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- \( \ldots \)
- Complex, rule-based configuration
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**Example**
Flight booking center: 100s of users, many parallel apps
**Example Rule**

```plaintext
correlation set/status/compound/satisfactory:
when status/compound NOT E {Satisfactory}
   AND status/startable E {Yes}
   AND ( ( status/observed E {Available, WasAvailable}
             AND status/desired E {Available}
             AND status/automation E {Idle, Internal}
             AND correlation/external/stop/failed E {false}
         OR
         ( status/observed E {SoftDown, StandBy}
             AND status/desired E {Unavailable}
             AND status/automation E {Idle, Internal}
         )
   )
then SetVariable status/compound = Satisfactory
   RecordVariableHistory status/compound
```

Fig. 4. Example of a correlation rule.

(taken from [SinzEtAl02])
Rules

when cond then var = d

- **AND, OR, NOT** allowed in conditions
- \( var \in \{ d_1, \ldots, d_2 \} \) – “element of”
- the **then** part can be executed if **cond** is true
Logical Encoding

- One boolean atom per var/value-pair

\[ \text{Encode that } \text{var} \text{ has exactly one value (of } d_1, \ldots, d_k) \]

\[ ( \bigvee_{i=1}^{k} \varphi_{\text{var}, d_i} ) \land \left( \bigwedge_{i,j=1}^{k} i < j \neg ( \varphi_{\text{var}, d_i} \land \varphi_{\text{var}, d_j} ) \right) \]

**Atomic Actions**: \( \text{var} = d \mapsto \alpha \)

**Axiom \[ \alpha_{\text{var}, d} \] \[ \varphi_{\text{var}, d} \]**
Logical Encoding

- One boolean atom per var/value-pair
- \( P_{\text{var}, d} = \text{true} \iff \text{var} = d \)
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- Atomic Actions: \( \text{var} = d \leadsto \alpha_{\text{var},d} \)
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- One boolean atom per var/value-pair
- \( P_{var,d} = true \iff var = d \)

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\]

- Atomic Actions: \( var = d \leadsto \alpha_{var,d} \)
- Axiom \([\alpha_{var,d}] P_{var,d}\)
Logical Encoding

Semantics of a rule as program:

\( ?\text{when} \); \( \text{then} \)
Logical Encoding

Semantics of a rule as program:

\[ \text{?when } ; \text{then} \]

Semantics of all rules as program:

\[ R := (\text{?when}_1 ; \text{then}_1) \cup \ldots \cup (\text{?when}_r ; \text{then}_r)^* \]
Proof Obligations

**Uniqueness of final state:**
under assumption of a precondition \( PRE \)

\[
PRE \rightarrow (\langle R \rangle p \leftrightarrow [R]p)
\]
Proof Obligations

**Uniqueness of final state:**
under assumption of a precondition $PRE$

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**Confluence:**

$$PRE \rightarrow (\langle R \rangle[R]p \rightarrow [R]\langle R \rangle p)$$
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**Absence of Oscillation:**
modelled using an extension of PDL with non-termination operator
Verification Experiment

Verification Technique

- state vector semantics
- translation of PDL to boolean SAT
- solving using SAT solver (Davies-Putnam)

Experiment:

- ∼40 rules
- resulted in ∼1500 boolean variables
- SAT solving < 1 sec

!! violations found – before deployment