Wahlen sind wichtig!

Karlsruher Institut für Technologie

> “*Alle Staatsgewalt geht vom Volke aus.*”
>
> *Art. 20 Abs. 2 GG*

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KIT
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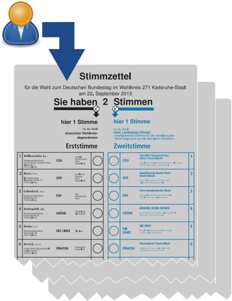
Formale Beweise!

Wahlablauf


Stimmzettel
für die Wahl zum Deutschen Bundestag im Wahlkreis 271 Karlsruhe-Stadt
am 22. September 2013
Sie haben 2 Stimmen
hier 1 Stimme
Erststimme
hier 1 Stimme
Zweitstimme
CDU
SPD
FDP
GRÜNE
DIE LINKE
PIRATEN

Wahlablauf



Endgültiges Ergebnis der Bundestagswahl 2013

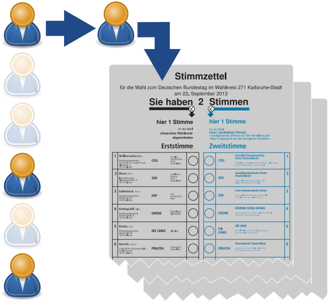
Gegenstand der Nachweisung	Erststimmen Anzahl	Zweitstimmen Anzahl
Wahlberechtigte	61.946.900	61.946.900
Wähler	44.309.925	44.309.925
Ungültige	684.883	583.069
Gültige	43.625.042	43.726.856
CDU	16.233.642	14.921.877
SPD	12.843.458	11.252.215
FDP	1.028.645	2.083.533
DIE LINKE	3.585.178	3.755.699
GRÜNE	3.180.299	3.694.057

BWahlG (~2 Seiten)

Zählung der Stimmen

Ermittlung des Ergebnisses

Wahlablauf



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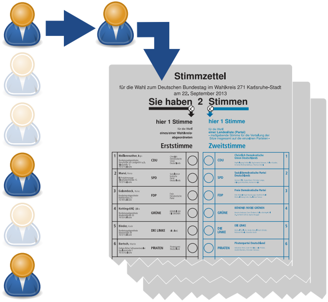

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Ermittlung des Ergebnisses

Teilaspekte

- Wählerregistrierung
- Stimmabgabe
- Übermittlung der Stimmen
- Zählung der Stimmen
- Ermittlung des Ergebnisses
- …

A Satz von Arrow

A Satz von Arrow

Universalität / Totalität

A Satz von Arrow

Universalität / Totalität

\+ Surjektivität

A Satz von Arrow

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\+ Surjektivität
\+ Monotonie

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+ Monotonie
+ Unabhängigkeit von irrelevanten Alternativen

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\+ Unabhängigkeit von irrelevanten Alternativen

= **Diktatur**

A Eigenschaften in der Praxis

Eigenschaften immer ein Kompromiss

	Majority (MC)	Majority loser	Mutual MC	Condorcet	Cond. loser	Smith/ ISDA	LIIA	IIA	Clone-proof	Mono-tone	Consistency	Participation	Reversal symmetry	Polytime/ Resolvable		Summable
Approval	Rated [nb 1]	No	No	No [nb 2][nb 3]	No	No [nb 2]	Yes	Yes [nb 4]	Yes [nb 5]	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)
Borda count	No	Yes	No	No [nb 2]	Yes	No	No	No	No: teams	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)
Copeland	Yes	Yes	Yes	Yes	Yes	Yes	No	No [nb 2]	teams, crowds	Yes	No [nb 2]	No [nb 2]	Yes	O(N²)	No	O(N²)
IRV (AV)	Yes	Yes	Yes	No [nb 2]	Yes	No [nb 2]	No	No	Yes	No	No	No	No	O(N²)	Yes	O(N!) [nb 7]
Kemeny-Young	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No [nb 2]	No: spoil-ers	Yes	No [nb 2] [nb 8]	No [nb 2]	Yes	O(N!)	Yes	O(N²) [nb 9]
Majority Judgment [nb 10]	Rated [nb 11]	No	No [nb 12]	No [nb 2][nb 3]	No	No [nb 2]	Yes	Yes [nb 4]	Yes	Yes	No [nb 13]	No [nb 14]	Depends [nb 15]	O(N)	Yes	O(N) [nb 16]
Minimax	Yes	No	No	Yes [nb 19]	No	No	No	No [nb 2]	No: spoil-ers	Yes	No [nb 2]	No [nb 2]	No	O(N²)	Yes	O(N²)

en.wikipedia.org/wiki/Voting_system

Cornerstones in the History of Social Choice Theory
Llull

- Ramon Llull (ca. 1232 – 1315/16).
- Writer, philosopher, poet, logician, martyr, founder of Catalan literature and author of *Blanquerna* (1283).
  - *Artifitium electionis personarum* (rediscovered 1959)
  - *Blanquerna, Ch. 24, En qual manera Natana fo eleta abadessa*
  - *De arte eleccionis* (1299, rediscovered 1937)

Cornerstones in the History of Social Choice Theory

Cusanus

- Nikolaus von Kues (Nicolaus Cusanus) (1401 – 1464).
- Philosopher, theologian, jurist, mathematician, astronomer.
    - In *De concordantia catholica* (1433) he devised a method to elect the Holy Roman Emperors. Each voter assigns points to candidates, and the candidate with highest score wins.

Cornerstones in the History of Social Choice Theory

Borda

- Jean-Charles Chevalier de Borda (1733 – 1799).
- Mathematician, military engineer and sailor.
    - In 1770 Borda reinvented the method of Nikolaus von Kues and suggested it for use in the French Academy of Sciences. The method is known today as the *Borda count*.

Borda count vs. plurality rule

Example

4 voters	3 voters	2 voters	**"Borda" scores**
a	b	c	**3**
b	c	b	**2**
c	a	a	**1**

- Candidate a is the plurality winner (he gets 4 votes as compared to 3 votes for b and 2 votes for c). But a is the worst candidate for a majority of 5 voters!
- According to Borda's method b is the winner with $4 \cdot 2 + 3 \cdot 3 + 2 \cdot 2 = 21$ points as compared to $4 \cdot 3 + 3 \cdot 1 + 2 \cdot 1 = 17$ points for a and $4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 = 16$ points for c.

- Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet (1743 – 1794).
- Philosopher, mathematician and political scientist.
  - *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* (1785).

Borda Count vs. Condorcet Winner

An alternative is called a *Condorcet winner* if it beats every other alternative in binary comparison by a majority of votes.

Example

3 voters	2 voters	"Borda" scores
a	b	**3**
b	c	**2**
c	a	**1**

- Candidate b is the Borda winner (with $3 \cdot 2 + 2 \cdot 3$ points) while a is the Condorcet winner.
- Moreover, c is *unanimously* deemed inferior to b. But if c is deleted, a is the unambiguous winner in the remaining problem to choose from $\{a, b\}$. In particular, a is the Borda winner in the reduced problem.

Voting Procedures and their Properties

Voting Procedures

We'll discuss procedures for n *voters* (or *individuals*, *agents*, *players*) to collectively choose from a set of m *alternatives* (or *candidates*):

- Each voter votes by submitting a *ballot*, e.g., the name of a single alternative, a ranking of all alternatives, or something else.
- The procedure defines what are *valid ballots*, and how to *aggregate* the ballot information to obtain a winner.

Remark 1: There could be *ties*. So our voting procedures will actually produce *sets of winners*. Tie-breaking is a separate issue.

Remark 2: Formally, *voting rules* (or *resolute* voting procedures) return single winners; *voting correspondences* return sets of winners.

Plurality Rule

Under the *plurality rule* each voter submits a ballot showing the name of one alternative. The alternative(s) receiving the most votes win(s).

Remarks:

- Also known as the *simple majority* rule ($\neq$ *absolute majority* rule).
- This is the most widely used voting procedure in practice.
- If there are only two alternatives, then it is a very good procedure.

Criticism of the Plurality Rule

Problems with the plurality rule (for more than two alternatives):

- The information on voter preferences other than who their favourite candidate is gets ignored.
- Dispersion of votes across ideologically similar candidates.
- Encourages voters not to vote for their true favourite, if that candidate is perceived to have little chance of winning.

Plurality with Run-Off

Under the *plurality rule with run-off*, each voter initially votes for one alternative. The winner is elected in a second round by using the plurality rule with the two top alternatives from the first round.

Remarks:

- Used to elect the president in France.
- Addresses some of the noted problems: elicits more information from voters; realistic "second best" candidate gets another chance.
- Still: heavily criticised after Le Pen entered the run-off in 2002.

The No-Show Paradox

Under plurality with run-off, it may be better to abstain than to vote for your favourite candidate! Example:

$$\begin{array}{ll} \text{25 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

Given these voter preferences, B gets eliminated in the first round, and C beats A 70:25 in the run-off.

Now suppose two voters from the first group abstain:

$$\begin{array}{ll} \text{23 voters:} & A \succ B \succ C \\ \text{46 voters:} & C \succ A \succ B \\ \text{24 voters:} & B \succ C \succ A \end{array}$$

A gets eliminated, and B beats C 47:46 in the run-off.

Borda Rule

Under the voting procedure proposed by Jean-Charles de Borda, each voter submits a complete ranking of all m candidates.

For each voter that places a candidate first, that candidate receives $m-1$ points, for each voter that places her 2nd she receives $m-2$ points, and so forth. The *Borda count* is the sum of all the points.

The candidate with the highest Borda count wins.

Remarks:

- Takes care of some of the problems identified for plurality voting, e.g., this form of balloting is more informative.
- Disadvantage (of any system where voters submit full rankings): higher elicitation and communication costs

J.-C. de Borda. *Mémoire sur les élections au scrutin*. Histoire de l'Académie Royale des Sciences, Paris, 1781.

Example

Consider again this example:

49%:	Bush $\succ$ Gore $\succ$ Nader
20%:	Gore $\succ$ Nader $\succ$ Bush
20%:	Gore $\succ$ Bush $\succ$ Nader
11%:	Nader $\succ$ Gore $\succ$ Bush

Our voting procedures give different winners:

- Plurality: Bush wins
- Plurality with run-off: Gore wins (Nader eliminated in round 1)
- Borda: Gore wins ($49 + 40 + 40 + 11 > 98 + 20 > 20 + 22$)
- Gore is also the *Condorcet winner* (wins any pairwise contest).

Positional Scoring Rules

We can generalise the idea underlying the Borda rule as follows:

A *positional scoring rule* is given by a *scoring vector* $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$.

Each voter submits a ranking of the m alternatives. Each alternative receives s_i points for every voter putting it at the ith position.

The alternative with the highest score (sum of points) wins.

Remarks:

- The *Borda rule* is is the positional scoring rule with the scoring vector $\langle m-1, m-2, \ldots, 0 \rangle$.
- The *plurality rule* is the positional scoring rule with the scoring vector $\langle 1, 0, \ldots, 0 \rangle$.

The Condorcet Principle

An alternative that beats every other alternative in pairwise majority contests is called a *Condorcet winner*.

There may be no Condorcet winner; witness the *Condorcet paradox:*

Ann:	$A \succ B \succ C$
Bob:	$B \succ C \succ A$
Cesar:	$C \succ A \succ B$

Whenever a Condorcet winner exists, then it must be *unique*.

A voting procedure satisfies the *Condorcet principle* if it elects (only) the Condorcet winner whenever one exists.

M. le Marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilté des décisions rendues a la pluralité des voix*. Paris, 1785.

Positional Scoring Rules violate Condorcet

Consider the following example:

3 voters: $A \succ B \succ C$
2 voters: $B \succ C \succ A$
1 voter: $B \succ A \succ C$
1 voter: $C \succ A \succ B$

A is the *Condorcet winner*; she beats both B and C $4:3$. But any *positional scoring rule* assigning strictly more points to a candidate placed 2nd than to a candidate placed 3rd ($s_2 > s_3$) makes B win:

$$A: \quad 3 \cdot s_1 + 2 \cdot s_2 + 2 \cdot s_3$$
$$B: \quad 3 \cdot s_1 + 3 \cdot s_2 + 1 \cdot s_3$$
$$C: \quad 1 \cdot s_1 + 2 \cdot s_2 + 4 \cdot s_3$$

Thus, *no positional scoring rule* (with a strictly descending scoring vector) will satisfy the *Condorcet principle*.

Copeland Rule

Some voting procedures (with ballots that are full rankings) have been designed specifically to meet the Condorcet principle.

The *Copeland rule* elects the alternative(s) that maximise(s) the difference between won and lost pairwise majority contests.

Remarks:

- The Copeland rule satisfies the Condorcet principle.
- Variations are possible: 0 points for every lost contest; 1 point for every won contest; α points (with possibly $\alpha \neq \frac{1}{2}$) for every draw

A.H. Copeland. *A "Reasonable" Social Welfare Function*. Seminar on Mathematics in Social Sciences, University of Michigan, 1951.

Tournament Solutions

The Copeland rule is an example for a *tournament solution*. There is an entire class of voting procedure that can be defined like this:

- Draw a directed graph where the alternatives are the vertices and there is an edge from A to B iff A beats B in a majority contest.

Many rules can be defined on such a *majority graph* (Laslier, 1997).

J.F. Laslier. *Tournament Solutions and Majority Voting*. Studies in Economic Theory, Springer-Verlag, 1997.

Kemeny Rule

Under the *Kemeny rule*, ballots are full rankings of the alternatives. An alternative wins if it is maximal in a ranking minimising the sum of disagreements with the ballots regarding pairs of alternatives.

That is:

(1) For every possible ranking R, count the number of triples (i, x, y) s.t. R disagrees with voter i on the ranking of alternatives x and y.

(2) Find all rankings R that have minimal score in the above sense.

(3) Elect any alternative that is maximal in such a "closest" ranking.

Remarks:

- Satisfies the Condorcet principle.
- This will be hard to compute (more later).

J. Kemeny. Mathematics without Numbers. *Daedalus*, 88:571–591, 1959.

Voting Trees (Cup Rule, Sequential Majority)

If ballots are rankings, we can define a voting rule via a *binary tree*, with the alternatives labelling the leaves, and an alternative progressing to a parent node if it beats its sibling in a *majority contest*.

Two examples for such rules and a possible profile of ballots:

```
(1)
            o
           / \
          o   C
         / \
        A   B
```

```
(2)       o
         / \
        /   \
       o     o
      / \   / \
     A   B B   C
```

$\mathtt{A} \succ \mathtt{B} \succ \mathtt{C}$

$\mathtt{B} \succ \mathtt{C} \succ \mathtt{A}$

$\mathtt{C} \succ \mathtt{A} \succ \mathtt{B}$

Rule (1): C wins

Rule (2): A wins

<u>Remarks:</u>

- Any such rule satisfies the Condorcet principle.
- Most such rules violate *neutrality* (= symmetry wrt. alternatives).

Single Transferable Vote (STV)

Also known as the *Hare system*. To select a single winner, it works as follows (voters submit ranked preferences for all candidates):

- If one of the candidates is the 1st choice for over 50% of the voters (*quota*), she wins.
- Otherwise, the candidate who is ranked 1st by the fewest voters gets *eliminated* from the race.
- Votes for eliminated candidates get *transferred:* delete removed candidates from ballots and "shift" rankings (i.e., if your 1st choice got eliminated, then your 2nd choice becomes 1st).

In practice, voters need not be required to rank all candidates (non-ranked candidates are assumed to be ranked lowest).

STV (suitably generalised) is often used to elect committees.

STV is used in several countries (e.g., Australia, New Zealand, . . .).

Example

Elect one winner amongst four candidates, using STV (100 voters):

39 voters:	$A \succ B \succ C \succ D$
20 voters:	$B \succ A \succ C \succ D$
20 voters:	$B \succ C \succ A \succ D$
11 voters:	$C \succ B \succ A \succ D$
10 voters:	$D \succ A \succ B \succ C$

(Answer: B wins)

Note that for 3 candidates, STV reduces to plurality voting with run-off, so it suffers from the same problems.

Single Transferable Vote

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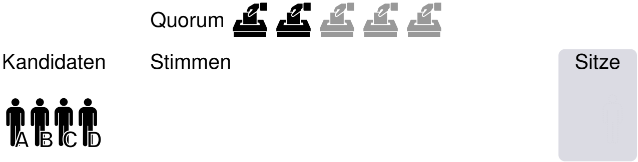
Quorum $\quad Q := \left\lfloor \frac{\textit{Stimmen}}{\textit{Sitze}+1} \right\rfloor + 1$

Wiederhole bis alle Sitze vergeben:

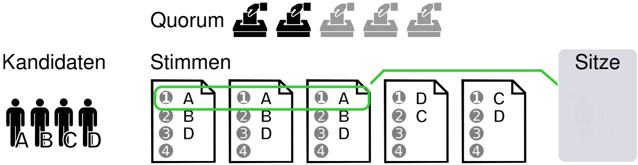
Gibt es einen Kandidaten K, der Q Erstpräferenzen erhält?

- Ja:
  - K ist gewählt
  - Q der K-Stimmzettel werden entfernt
  - entferne K von allen Stimmzetteln
- Nein:
  - entferne schwächsten Kandidaten von allen Stimmzetteln

Beispiel einer Wahl mit STV

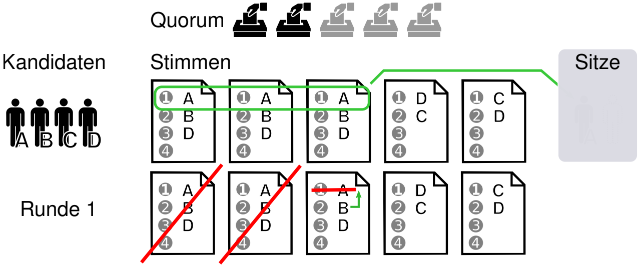


Beispiel einer Wahl mit STV

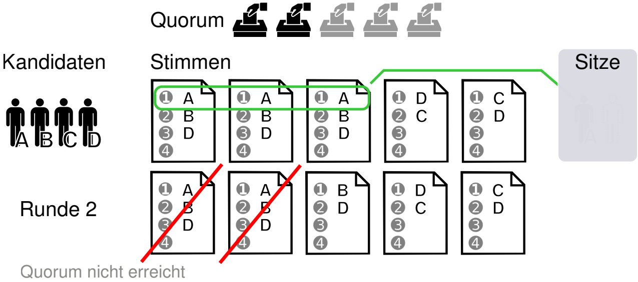


Runde 1

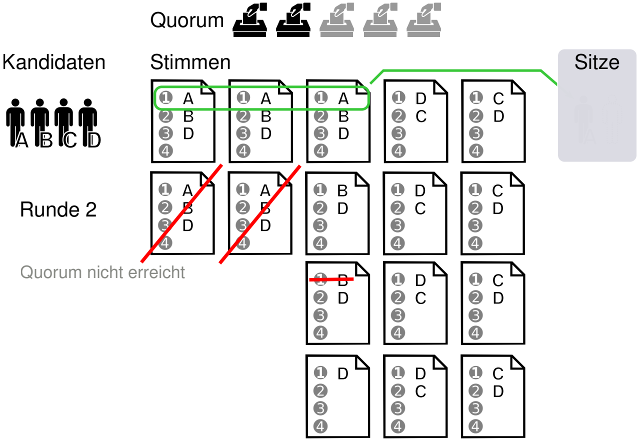
Beispiel einer Wahl mit STV



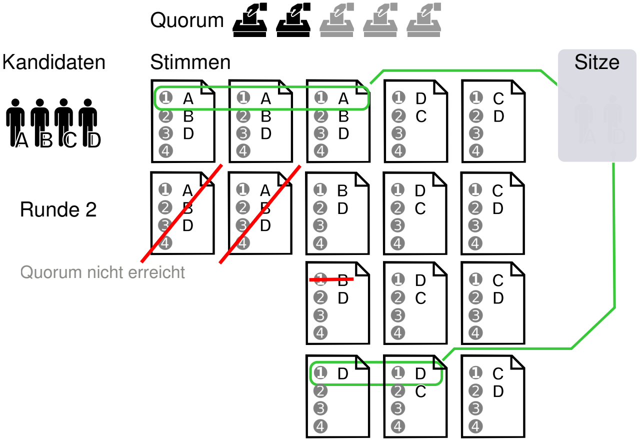
Beispiel einer Wahl mit STV



Beispiel einer Wahl mit STV



Beispiel einer Wahl mit STV



Approval Voting (AV)

In *approval voting*, a ballot is a *set* of alternatives (the ones the voter "approves" of). The alternative with the most approvals wins.

Remarks:

- Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).
- Intuitively, less cause *not* to vote for the most preferred candidate for strategic reasons when she has a slim chance of winning.
- Good compromise between plurality (too simple) and Borda (too complex) in terms of communication requirements.
- Only procedure we have seen where ballots *cannot* be modelled as linear orders over the set of alternatives.

S.J. Brams and P.C. Fishburn. Approval Voting. *The American Political Science Review*, 72(3):831–847, 1978.

Summary: Voting Procedures

We have seen a fair number of voting *procedures:*

- *Ballots* might be elements (plurality), rankings (e.g., Borda), or subsets (approval) of the set of alternatives. (Enough for AI?)
- Types of procedures:
  - *positional scoring rules:* Borda, (plurality)
  - based on the *majority graph:* Copeland, voting trees
  - based on the *weighted majority graph:* Kemeny
  - *staged procedures:* plurality with run-off, STV
  - *approval voting*

We have seen a few *properties* of voting procedures:

- *Monotonicity*, as violated by e.g. the *no-show paradox*
- *Strategic* issues, meaning people might not vote truthfully
- *Condorcet principle:* if an alternative wins all pairwise majority contests, then it should win the election

Major Theorems in Voting Theory

The Axiomatic Method

Most of the important classical results in voting theory are *axiomatic*. They formalise desirable properties as "axioms" and then establish:

- *Characterisation Theorems*, showing that a particular (class of) procedure(s) is the only one satisfying a given set of axioms
- *Impossibility Theorems*, showing that there exists *no* voting procedure satisfying a given set of axioms

We will see two examples each (+ one other thing).

Formal Framework

Basic terminology and notation:

- finite set of *voters* $\mathcal{N} = \{1, \ldots, n\}$, the *electorate*
- (usually finite) set of *alternatives* $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$
- Denote the set of *linear orders* on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$. *Preferences* are assumed to be elements of $\mathcal{L}(\mathcal{X})$. *Ballots* are elements of $\mathcal{L}(\mathcal{X})$.

A *voting procedure* is a function $F : \mathcal{L}(\mathcal{X})^n \to 2^{\mathcal{X}} \setminus \{\emptyset\}$, mapping profiles of ballots to nonempty sets of alternatives.

Remark: AV does not fit in this framework; everything else does.

Two Alternatives

When there are only *two alternatives*, then all the voting procedures we have seen coincide, and *intuitively* they do the “right” thing.

Can we make this intuition precise?

▶ Yes, using the axiomatic method.

Anonymity

A voting rule is *anonymous* if the *voters* are treated symmetrically: if two voters switch ballots, then the winners don't change.

Formally:

> F is anonymous if $F(b_1, \ldots, b_n) = F(b_{\pi(1)}, \ldots, b_{\pi(n)})$ for any ballot profile $(b_1, \ldots, b_n)$ and any permutation $\pi : \mathcal{N} \to \mathcal{N}$.

Neutrality

A voting procedure is *neutral* if the *alternatives* are treated symmetrically.

Formally:

> F is neutral if $F(\pi(\underline{b})) = \pi(F(\underline{b}))$ for any ballot profile $\underline{b}$ and any permutation $\pi : \mathcal{X} \to \mathcal{X}$ (with π extended to ballot profiles and sets of alternatives in the natural manner).

Positive Responsiveness

A voting procedure satisfies the property of *positive responsiveness* if, whenever some voter raises a (possibly tied) winner x in her ballot, then x will become the unique winner.

Formally:

> F satisfies positive responsiveness if $x \in F(\underline{b})$ implies $\{x\} = F(\underline{b}')$ for any alternative x and any two distinct profiles $\underline{b}$ and $\underline{b}'$ with $\underline{b}(x \succ y) \subseteq \underline{b}'(x \succ y)$ and $\underline{b}(y \succ z) = \underline{b}'(y \succ z)$ for all alternative y and z different from x.

<u>Notation:</u> $\underline{b}(x \succ y)$ is the set of voters ranking x above y in $\underline{b}$

May's Theorem

Now we can fully characterise the plurality rule:

Theorem 1 (May, 1952) *A voting procedure for two alternatives satisfies anonymity, neutrality, and positive responsiveness if and only if it is the plurality rule.*

Remark: In these slides we assume that there are no indifferences in ballots, but May's Theorem also works (with an appropriate definition of positive responsiveness) when ballots are weak orders.

K.O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decisions. *Econometrica*, 20(4):680–684, 1952.

Proof Sketch

Clearly, plurality does satisfy all three properties. ✓

Now for the other direction:

For simplicity, assume the number of voters is odd (no ties).
Plurality-style ballots are fully expressive for two alternatives.
Anonymity and neutrality $\rightsquigarrow$ only number of votes matters.

Denote as A the set of voters voting for alternative a and as B those voting for b. Distinguish two cases:

- Whenever $|A| = |B| + 1$ then only a wins. Then, by PR, a wins whenever $|A| > |B|$ (that is, we have plurality). ✓
- There exist A, B with $|A| = |B| + 1$ but b wins. Now suppose one a-voter switches to b. By PR, now only b wins. But now $|B'| = |A'| + 1$, which is symmetric to the earlier situation, so by neutrality a should win $\rightsquigarrow$ contradiction. ✓

Characterisation Theorems

When there are more than two alternatives, then different voting procedures are really different. To choose one, we need to understand its properties: ideally, we get a *characterisation theorem*.

Maybe the best known result of this kind is Young's characterisation of the *positional scoring rules* (PSR) ...

<u>Reminder:</u>

- Every scoring vector $s = \langle s_1, \ldots, s_m \rangle$ with $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$ defines a PSR: give s_i points to alternative x whenever someone ranks x at the ith position; the winners are the alternatives with the most points.

Reinforcement (a.k.a. Consistency)

A voting procedure satisfies *reinforcement* if, whenever we split the electorate into two groups and some alternative would win in both groups, then it will also win for the full electorate.

For a full formalisation of this concept we would need to be able to speak about a voting procedure F wrt. different electorates $\mathcal{N}$, $\mathcal{N}'$, ...

Formally (under natural refinements to our notation):

> F satisfies reinforcement if $F^{\mathcal{N} \cup \mathcal{N}'}(\underline{b}) = F^{\mathcal{N}}(\underline{b}) \cap F^{\mathcal{N}'}(\underline{b})$ for any disjoint electorates $\mathcal{N}$ and $\mathcal{N}'$ and any ballot profile $\underline{b}$ such that $F^{\mathcal{N}}(\underline{b}) \cap F^{\mathcal{N}'}(\underline{b}) \neq \emptyset$.

Continuity

A voting procedure is *continuous* if, whenever electorate $\mathcal{N}$ elects a unique winner x, then for any other electorate $\mathcal{N}'$ there exists a number k s.t. $\mathcal{N}'$ together with k copies of $\mathcal{N}$ will also elect only x.

Young's Theorem

We are now ready to state the theorem:

Theorem 2 (Young, 1975) *A voting procedure satisfies anonymity, neutrality, reinforcement, and continuity iff it is a positional scoring rule.*

Proof: Omitted (and difficult).

But it is not hard to verify the right-to-left direction.

H.P. Young. Social Choice Scoring Functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.

Impossibility Theorems

Another important type of result are *impossibility theorems:*

- showing that a certain combination of axioms is *inconsistent*
- alternative reading: a certain set of axioms *characterises* an obviously unattractive rule (directly violating a final axiom)

We first discuss Arrow' Theorem . . .

Unanimity and the Pareto Condition

A voting procedure is *unanimous* if it elects only x whenever all voters say that x is the best alternative. Formally:

> F is unanimous if whenever $\underline{b}(x \succ y) = \mathcal{N}$ for all $y \in \mathcal{N} \setminus \{x\}$ then $F(\underline{b}) = \{x\}$.

The *weak Pareto condition* is slightly less demanding. It is satisfied if an alternative y that is dominated by some other alternative x in all ballots cannot win. Formally:

> F is weakly Pareto if $\underline{b}(x \succ y) = \mathcal{N}$ implies $y \notin F(\underline{b})$.

Independence of Irrelevant Alternatives (IIA)

A voting procedure is *irrelevant of independent alternatives* if, whenever y loses to some winner x and the relative ranking of x and y does not change in the ballots, then y cannot win (independently of any possible changes wrt. other, irrelevant, alternatives).

Formally:

> F satisfies IIA if $x \in F(\underline{b})$ and $y \notin F(\underline{b})$ together with $\underline{b}(x \succ y) = \underline{b'}(x \succ y)$ imply $y \notin F(\underline{b'})$ for any profiles $\underline{b}$ and $\underline{b'}$.

Remark: This variant if IIA (for voting rules) is due to Taylor (2005). Arrow's original formulation of IIA is for *social welfare functions*, where the outcome is a preference ordering.

A.D. Taylor. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press, 2005.

K.J. Arrow. *Social Choice and Individual Values*. 2nd edition. Cowles Foundation, Yale University Press, 1963.

Dictatorships

- A voting procedure is a *dictatorship* if there exists a voter such that the unique winner will always be the top-ranked alternative of that voter (the dictator).
- A voting procedure is *nondictatorial* if it is not a dictatorship.

Arrow's Theorem for Voting Procedures

This is widely regarded as *the* seminal result in Social Choice Theory. Kenneth J. Arrow received the Nobel Prize in Economics in 1972.

Theorem 3 (Arrow, 1951) *No voting procedure for $\geqslant 3$ alternatives is weakly Pareto, IIA, and nondictatorial.*

Proof: Omitted.

This particular version of the theorem is proved by Taylor (2005).

Maybe the most accessible proof (of the standard formulation of the theorem) is the first proof in the paper by Geanakoplos (2005).

K.J. Arrow. *Social Choice and Individual Values*. 2nd edition. Cowles Foundation, Yale University Press, 1963.

A.D. Taylor. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press, 2005.

J. Geanakoplos. Three Brief Proofs of Arrow's Impossibility Theorem. *Economic Theory*, 26(1):211–215, 2005.

Remarks

- Note that this is a *surprising* result!
- Note that the theorem does *not* hold for *two* alternatives.
- We can interpret the theorem as a *characterisation* result:

  A voting procedure for $\geqslant 3$ alternatives satisfies the weak Pareto condition and IIA if and only if it is a dictatorship.

- *IIA* is the most debatable of the three axioms featuring in the theorem. Indeed, it is quite hard to satisfy.

Manipulation

Let's look once more at our favourite example:

49%:	Bush $\succ$ Gore $\succ$ Nader
20%:	Gore $\succ$ Nader $\succ$ Bush
20%:	Gore $\succ$ Bush $\succ$ Nader
11%:	Nader $\succ$ Gore $\succ$ Bush

Under the plurality rule, the Nader supporters could *manipulate:* pretend they like Gore best and improve the result.

Ideally, there would be no need for voters to strategise in this way.
Ideally, we'd like a procedure that is *strategy-proof* .

Strategy-Proofness

Recall: F is *resolute* if $F(\underline{b})$ is a singleton for any profile of ballots $\underline{b}$.

Let $\succ_i$ be the true preference of voter i and let b_i be the ballot of i.

A resolute voting procedure is *strategy-proof* if there exist no profile $\underline{b} = (b_1, \ldots, b_n)$ and no voter i s.t. $F(\underline{b}) \succ_i F(b_1, \ldots, \succ_i, \ldots, b_n)$, with $\succ_i$ lifted from alternatives to singletons in the natural manner.

The Gibbard-Satterthwaite Theorem

A resolute voting procedure F is *surjective* if for any alternative x there exists a ballot profile $\underline{b}$ such that $F(\underline{b}) = \{x\}$.

Theorem 4 (Gibbard-Satterthwaite) *Any resolute voting procedure for $\geqslant 3$ alternatives that is surjective and strategy-proof is dictatorial.*

Remarks:

- Again, *surprising*. Again, not applicable for *two* alternatives.
- The opposite direction is clear: *dictatorial* $\Rightarrow$ *strategy-proof*
- *Random* procedures don't count (but might be "strategy-proof").

A. Gibbard. Manipulation of Voting Schemes: A General Result. *Econometrica*, 41(4):587–601, 1973.

M.A. Satterthwaite. Strategy-proofness and Arrow's Conditions. *Journal of Economic Theory*, 10:187–217, 1975.

Proof Sketch

One way of proving this involves the notion of a *pivotal voter*.
Benoît (2000) gives a simple proof based on this idea.

The main steps are:

- show that when all voters rank x last, then x doesn't win
- show that when all voters rank x on top, then x wins
- observe that when we let voters switch x from bottom to top one by one, there must be a *pivotal voter* i causing x to win
- show that i can dictate x's relative position wrt. any y
- repeat this for every alternative: each has a "local" dictator
- observe that, by definition, there can be only one dictator

J.-P. Benoît. The Gibbard-Satterthwaite Theorem: A Simple Proof. *Economic Letters*, 69:319–322, 2000.

Domain Restrictions

How can we circumvent these impossibilities?

- Note that we have made an implicit *universal domain* assumption: *any* linear order may come up as a preference or ballot.
- If we *restrict* the domain (possible ballot profiles + possible preferences), more procedures will satisfy more axioms ...

Single-Peaked Preferences

An electorate $\mathcal{N}$ has *single-peaked* preferences if there exists a "left-to-right" ordering $\gg$ on the alternatives such that any voter prefers x to y if x is between y and her top alternative wrt. $\gg$.

The same definition can be applied to profiles of ballots.

Remarks:

- Quite natural: classical spectrum of political parties; decisions involving agreeing on a number (e.g., legal drinking age); . . .
- But certainly not universally applicable.

Black's Median Voter Theorem

For simplicity, assume the number of voters is *odd*.

For a given left-to-right ordering $\gg$, the *median voter rule* asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median wrt. $\gg$.

Theorem 5 (Black's Theorem, 1948) *If an odd number of voters submit single-peaked ballots, then there exists a Condorcet winner and it will get elected by the median voter rule.*

D. Black. On the Rationale of Group Decision-Making. *The Journal of Political Economy*, 56(1):23–34, 1948.

Proof Sketch

The candidate elected by the median voter rule is a Condorcet winner:

> Proof: Let x be the winner and compare x to some y to, say, the left of x. As x is the median, for more than half of the voters x is between y and their favourite, so they prefer x. ✓

Note that this also implies that a Condorcet winner exists.

As the Condorcet winner is (always) unique, it follows that, also, every Condorcet winner is a median voter rule election winner. ✓

Consequences

If the number of voters is odd and their preferences (and ballots) are single-peaked wrt. a known order, then:

- The median voter rule (= electing the Condorcet winner) is *strategy-proof* (Gibbard-Satterthwaite fails).
- The median voter rule (= electing the Condorcet winner) is *weakly Pareto* and *IIA* (Arrow fails).

Summary: Major Theorems

We have seen some of the major theorems in Social Choice Theory pertaining to voting, using the *axiomatic method:*

- *May:* plurality for two alternatives is characterised by anonymity, neutrality and positive responsiveness
- *Young:* positional scoring rules are characterised by reinforcement
- *Arrow:* Pareto (unanimity) and independence lead to dictatorships
- *Gibbard-Satterthwaite:* strategy-proofness leads to dictatorships
- *Black:* single-peakedness solves most problems

Other classics to look out for:

- *McGarvey:* any majority graph can occur
- *Sen:* impossibility of a Paretian liberal
- *Sen:* triple-wise value restriction, generalising single-peakedness
- *Duggan-Schwartz:* G-S for irresolute voting procedures
- *Clarke* and *Groves:* strategy-proofness for quasi-linear preferences

Fair Allocation of Goods

Consider a set of agents and a set of goods. Each agent has her own preferences regarding the allocation of goods to agents. Examples:

- allocation of resources amongst members of our society
- allocation of bandwith to processes in a communication network
- allocation of compute time to scientists on a super-computer
- ...

We will focus on one specific model studied in the literature, with a single good that can be divided into arbitrarily small pieces ...

Cake Cutting

A classical example for a problem of collective decision making:

> *We have to divide a cake with different toppings amongst n agents by means of parallel cuts. Agents have different preferences regarding the toppings (additive utility functions).*

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|----------------------|
0                      1
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The exact details of the formal model are not important for this short exposition. You can look them up in my lecture notes (cited below).

U. Endriss. *Lecture Notes on Fair Division*. Institute for Logic, Language and Computation, University of Amsterdam, 2009/2010.

Cut-and-Choose

The classical approach for dividing a cake between *two agents:*

- *One agent cuts the cake in two pieces (she considers to be of equal value), and the other chooses one of them (the piece she prefers).*

The cut-and-choose protocol is *fair* in the sense of guaranteeing a property known as *proportionality:*

- Each agent is *guaranteed* at least one half (general: $1/n$), according to her own valuation.
- Discussion: In fact, the first agent (if she is risk-averse) will receive exactly $1/2$, while the second will usually get more.

What if there are *more than two* agents?

The Banach-Knaster Last-Diminisher Protocol

In the first ever paper on fair division, Steinhaus (1948) reports on a *proportional* protocol for n *agents* due to Banach and Knaster.

(1) Agent 1 cuts off a piece (that she considers to represent $1/n$).

(2) That piece is passed around the agents. Each agent either lets it pass (if she considers it too small) or trims it down further (to what she considers $1/n$).

(3) After the piece has made the full round, the last agent to cut something off (the "last diminisher") is obliged to take it.

(4) The rest (including the trimmings) is then divided amongst the remaining $n-1$ agents. Play cut-and-choose once $n = 2$. ✓

Each agent is guaranteed a *proportional* piece. Requires $O(n^2)$ cuts. May not be *contiguous* (unless you always trim "from the right").

H. Steinhaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948.

Matching

In a variant of the fair allocation problem, we try to match each agent with a single item—which may have preferences itself. Examples:

- children to schools
- junior doctors to hospitals
- kidney patients to kidney donors
- ...

We now briefly look into *the* classical matching problem.

The Stable Marriage Problem

We are given:

- n *men* and n *women*
- each has a linear *preference* ordering over the opposite sex

We seek:

- a *stable* matching of men to women: no man and woman should want to divorce their assigned partners and run off with each other

The Gale-Shapley Algorithm

Theorem 1 (Gale and Shapley, 1962) *There exists a stable matching for any combination of preferences of men and women.*

The *Gale-Shapley "deferred acceptance" algorithm* for computing a stable matching works as follows:

- In each round, each man who is not yet engaged proposes to his favourite amongst the women he has not yet proposed to.
- In each round, each woman picks her favourite from the proposals she's receiving and the man she's currently engaged to (if any).
- Stop when everyone is engaged.

D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage. *American Mathematical Monthly*, 69:9–15, 1962.