Formal Systems II: Theory

Separation Logic

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Mattias Ulbrich
Institute of Theoretical Informatics
Motivation
Reminder . . . Dynamic Logic

Given: a program with a contract:

1. precondition, FOL formula $pre$
2. postcondition, FOL formula $post$
3. code, while program $\pi$

In program verification, one formally proves that

$$\mathbb{N} \models pre \rightarrow [\pi]post$$

If $pre$ holds before execution of $\pi$ then $post$ holds after termination.

Reminder: weakest precondition calculus for DL.
The Framing Problem

Formal Software Verification

- Prove what effects a program has.

Example (after McCarthy and Hayes, 1969)
P calls operator to ask for Q’s number.

Precondition: P has a telephone.

Postcondition: P knows the number of Q.

missing postcondition?

Postcondition: P still has a telephone.
The Framing Problem

Formal Software Verification

- Prove what effects a program has.
- Prove what effects a program does *not* have.
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You should not have to specify the latter explicitly.

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\( P \) calls operator to ask for \( Q \)'s number.

- **Precondition**: \( P \) has a telephone.
- **Postcondition**: \( P \) knows the number of \( Q \)
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Example in Java

```java
interface Account {
    void setBalance(int);
    int getBalance();
}
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The Framing Problem

Example in Java

```java
interface Account {
    void setBalance(int);
    int getBalance();
}

//@ ensures \result == 100;
int f(Account account1, Account account2) {
    account1.setBalance(100);
    account2.setBalance(200);
    return account1.getBalance();
}
```
interface Account {
    void setBalance(int);
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//@ requires account1 != account2;
//@ ensures \result == 100;
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- `setBalance` does not effect other accounts
- `setBalance` does not effect other customer objects
- `setBalance` does not effect any object of any classes which may be added later.
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The Framing Problem

Problem statement

In program verification, the framing problem is the problem to specify and verify that the effects of a program are limited to the data structure that is being operated on.

It is a challenge for the specifying user (needs to think about not-effects) and for reasoning engines (increased complexity).
The Framing Problem

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In program verification, the framing problem is the problem to specify and verify that the effects of a program are limited to the data structure that is being operated on. It is a challenge for the specifying user (needs to think about not-effects) and for reasoning engines (increased complexity).

Suggested solutions:

- Ownership (Types) (Noble, Vitek and Potter 1998)
- Separation Logic (Reynolds, 1999)
- Dynamic Frames/Region Logic (Kassios 2006)
- ...
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In program verification, the framing problem is the problem to specify and verify that the effects of a program are limited to the data structure that is being operated on.

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Suggested solutions:

- Ownership (Types)  (Noble, Vitek and Potter 1998)
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- ...
Heaps and “Footprints”
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Heap

account1

account2
Heaps and “Footprints”
Heaps and Heaplets

Modelling assumptions

- Every memory location holds a value in $\mathbb{N}$.
- There infinitely many memory locations.

Heap and Heaplet

A **heap** is a total function modelling memory:

$$heap : \mathbb{N} \rightarrow \mathbb{N}$$

A **heaplet** is a finite partial function modelling footprints:

$$heaplet : \mathbb{N} \mapsto \mathbb{N}$$

Partial function:

Partial function $f : A \mapsto B$ is a function $f : D \rightarrow B$ for $D \subseteq A$. The finite set $D = \text{dom } f$ is called the domain of $f$. 
Disjoint union of heaplets:
\[ h = h_1 \cup h_2 \text{ iff } \text{dom } h_1 \cap \text{dom } h_2 = \emptyset \text{ and } h = h_1 \cup h_2. \]

\( h_1 \cup h_2 \) is always a heaplet.
(Union \( \cup \) of heaplets does not always result in heaplets.)

Membership

For \((x, y) \in h\) write \(h(x) = y\).

It means: Memory location \(x\) holds value \(y\).

Empty Heap

The empty heaplet \(\emptyset\) is without allocated locations.

Singletons

Heaplet with exactly one allocated location \(x\) which holds value \(y\):
write \(h = \{(x, y)\}\)
Separation Logic
Separation Logic – Syntax

Terms $t$: 

new in Separation Logic
Separation Logic – Syntax

Terms $t$:

- FOL terms over $\mathbb{N}$ with $+, -, \cdot, 0, 1$

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Formulae $\varphi$:

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Formulae $\varphi$:
- $\varphi_1 \land \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 \rightarrow \varphi_2$

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Separation Logic – Syntax

Terms $t$:
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Formulae $\varphi$:
- $\varphi_1 \land \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 \to \varphi_2$
- $t_1 = t_2$, $t_1 < t_2$, ...

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- $\forall x. \varphi$, $\exists x. \varphi$

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- $\varphi_1 \ast \varphi_2$

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Separation Logic – Syntax

Terms \( t \):
- FOL terms over \( \mathbb{N} \) with \(+, -, \cdot, 0, 1\)

Formulae \( \varphi \):
- \( \varphi_1 \land \varphi_2 \), \( \varphi_1 \lor \varphi_2 \), \( \varphi_1 \implies \varphi_2 \)
- \( t_1 = t_2 \), \( t_1 < t_2 \), . . .
- \( \forall x. \varphi \), \( \exists x. \varphi \)
- \( \varphi_1 \ast \varphi_2 \)
- \text{emp}

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- $\varphi_1 \land \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 \Rightarrow \varphi_2$
- $t_1 = t_2$, $t_1 < t_2$, ...
- $\forall x.\varphi$, $\exists x.\varphi$
- $\varphi_1 \ast \varphi_2$
- $\text{emp}$
- $t_1 \mapsto t_2$

new in Separation Logic
Separation Logic – Syntax

**Terms $t$:**
- FOL terms over $\mathbb{N}$ with $+, -, \cdot, 0, 1$

**Formulae $\varphi$:**
- $\varphi_1 \land \varphi_2, \varphi_1 \lor \varphi_2, \varphi_1 \rightarrow \varphi_2$
- $t_1 = t_2, t_1 < t_2, \ldots$
- $\forall x. \varphi, \exists x. \varphi$
- $\varphi_1 \ast \varphi_2$
- $\text{emp}$
- $t_1 \triangleright t_2$
- $\varphi_1 \ast \ast \varphi_2$ (later)

new in Separation Logic
Operator Precedence

How are the implicit parentheses in
\[ B \rightarrow C \land D \lor A \times x \mapsto y \]
How are the implicit parentheses in
\[ B \rightarrow^* C \land D \lor A \rightarrow^* x \mapsto y \]?
Operator Precedence

How are the implicit parentheses in

\( B \rightarrow\star C \land D \lor A \star x \leftrightarrow y \)?

**Binding force:**

- \( \star \) binds like \( \land \)
- \( \rightarrow\star \) binds like \( \rightarrow, \lor \)
- \( \leftrightarrow \) binds like \( = \)

**Answer:**

\[
(B \rightarrow\star (C \land D)) \lor (A \star (x \leftrightarrow y))
\]

or

\[
B \rightarrow\star ((C \land D) \lor (A \star (x \leftrightarrow y)))
\]
Operator Precedence

How are the implicit parentheses in

\[ B \rightarrow* (C \land D) \lor A \rightarrow* (x \leftrightarrow y) \]?

**Binding force:**

- \( \rightarrow* \) binds like \( \lor \), \( \land \)
- \( \rightarrow \) binds like \( \rightarrow \), \( \lor \)
- \( \leftrightarrow \) binds like \( = \)

**Answer:**

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\]

or

\[
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\]

Add explicit parentheses when combining \( \lor, \rightarrow \) or \( \land, \rightarrow* \).
Separation Logic – Semantics

Structure

Fixed first order domain: \( \mathbb{N} \).
Terms and formulas are evaluated over:

1. Variable assignment \( \beta : \text{Var} \rightarrow \mathbb{N} \)
2. Heaplet \( h : \mathbb{N} \rightarrow \mathbb{N} \)
### Structure

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### Terms:

- \( \text{val}_\beta(t_1 + t_2) = \text{val}_\beta(t_1) +_{\mathbb{N}} \text{val}_\beta(t_2) \), same for “·”
- \( \text{val}_\beta(x) = \beta(x) \) for variable \( x \)

### Formulas in FOL:

- Operator \( \beta, h \models \) is as expected for \( \wedge, \vee, \rightarrow, \forall, \exists, <, = \).
- Example: \( \beta, h \models \varphi_1 \wedge \varphi_2 \) iff \( \beta, h \models \varphi_1 \) and \( \beta, h \models \varphi_2 \)
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\[ \beta, h \models \text{emp} \iff \text{dom } h = \emptyset \]
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- $\beta, h \models \text{emp}$ iff $\text{dom } h = \emptyset$
- $\beta, h \models t_1 \leftrightarrow t_2$ iff $h = \{(\text{val}_{\beta}(t_1), \text{val}_{\beta}(t_2))\}$
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- \( \beta, h \models \varphi_1 \ast \varphi_2 \iff \text{there exist heaplets } h_1, h_2 : \mathbb{N} \rightarrow \mathbb{N} \text{ with } \)}
## Separation Logic – Semantics

### Structure

Fixed first order domain: $\mathbb{N}$.

Terms and formulas are evaluated over:

1. **Variable assignment** $\beta : \text{Var} \rightarrow \mathbb{N}$
2. **Heaplet** $h : \mathbb{N} \rightarrow \mathbb{N}$

- $\beta, h \models \text{emp}$ iff $\text{dom}(h) = \emptyset$
- $\beta, h \models t_1 \mapsto t_2$ iff $h = \{(\text{val}_\beta(t_1), \text{val}_\beta(t_2))\}$
- $\beta, h \models \varphi_1 \ast \varphi_2$ iff there exist heaplets $h_1, h_2 : \mathbb{N} \rightarrow \mathbb{N}$ with
  1. $h = h_1 \cup h_2$ and
Separation Logic – Semantics

Structure

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- \( \beta, h \models t_1 \mapsto t_2 \quad \text{iff} \quad h = \{(\text{val}_\beta(t_1), \text{val}_\beta(t_2))\} \)
- \( \beta, h \models \varphi_1 \ast \varphi_2 \quad \text{iff} \quad \text{there exist heaplets } h_1, h_2 : \mathbb{N} \rightarrow \mathbb{N} \text{ with} \)
  1. \( h = h_1 \cup h_2 \text{ and} \)
  2. \( \beta, h_1 \models \varphi_1 \text{ and} \)
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1. \( h = h_1 \cup h_2 \) and
2. \( \beta, h_1 \models \varphi_1 \) and
3. \( \beta, h_2 \models \varphi_2 \)
Connector $\ast$ is called **Separating Conjunction**

$A \ast B$ has the following intuitive semantics:

$A \ast B$ is true
\[\iff\]
$A$ is true
and $B$ is true
and $A$ and $B$ refer to disjoint sets of memory locations.
### Properties of Separation Logic

<table>
<thead>
<tr>
<th>Idempotence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\models A \leftrightarrow A \land A$</td>
</tr>
</tbody>
</table>

(idempotence for $\land$)
Properties of Separation Logic

Idempotence

- $\models A \leftrightarrow A \land A$ (idempotency for $\land$)
- $\not\models A \leftrightarrow A \star A$ (idempotence also for $\star$).
## Properties of Separation Logic

### Idempotence

| $\models A \iff A \land A$ | (idempotence for $\land$) |
| $\models ? A \iff A \ast A$ | (idempotence also for $\ast$?) |

**NO! Counterexample:**

| $\not\models (7 \mapsto \rightarrow 3 \ast \mapsto \rightarrow 7 \not\mapsto \rightarrow 3) \rightarrow 3 \mapsto \rightarrow 6 \ast \mapsto \rightarrow 4 \ast \mapsto \rightarrow 7$ |
Properties of Separation Logic

**Idempotence**

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- $\not\models A \leftrightarrow A \ast A$  
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- **NO!** Counterexample:
  $\models \neg(7 \leftrightarrow 3 \rightarrow 7 \leftrightarrow 3 \ast 7 \leftrightarrow 3)$
Properties of Separation Logic

Idempotence

- $\models A \leftrightarrow A \land A$ (idempotence for $\land$)
- $\models A \leftrightarrow A \ast A$ (idempotence also for $\ast$ ?)
- NO! Counterexample:
  $\models \neg (7 \leftrightarrow 3 \rightarrow 7 \leftrightarrow 3 \ast 7 \rightarrow 3)$

Weakening

- $\models A \land B \rightarrow A$ (Weakening of conjunction)
- $\models A \ast B \rightarrow A$ (Weakening of separating conjunction?)
- NO! Counterexample:
  $\models \neg (7 \leftrightarrow 3 \ast 6 \rightarrow 4 \rightarrow 7 \leftrightarrow 3)$
Properties of Separation Logic

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### Properties of Separation Logic

#### Idempotence

- \( \models A \leftrightarrow A \land A \)  
  (idempotence for \( \land \))
- \( \nvDash A \leftrightarrow A * A \)  
  (idempotence also for \( * \))

**NO!** Counterexample:

\[ \models \neg (7 \mapsto 3 \rightarrow 7 \mapsto 3 * 7 \mapsto 3) \]

#### Weakening

- \( \models A \land B \rightarrow A \)  
  (Weakening of conjunction)
- \( \nvDash A * B \rightarrow A \)  
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**NO!** Counterexample:
Properties of Separation Logic

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Weakening

- \( \models A \land B \rightarrow A \)  
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  (Weakening of separating conjunction?)
- NO! Counterexample:
  \( \models \neg (7 \leftrightarrow 3 \ast 6 \leftrightarrow 4 \rightarrow 7 \leftrightarrow 3) \)
\[ \beta, h \models A \leftrightarrow B \] means that:

- \[ \{ (\text{val}(A), \text{val}(B)) \} = h, \]
- not only \( (\text{val}(A), \text{val}(B)) \in h \)
Caution

$\beta, h \models A \leftrightarrow B$ means that:
- $\{(\text{val}(A), \text{val}(B))\} = h$,
- not only $(\text{val}(A), \text{val}(B)) \in h$

On the other hand:

$\beta, h \models ? \iff (\text{val}(A), \text{val}(B)) \in h$
Caution

\[ \beta, h \models A \leftrightarrow B \text{ means that:} \]

1. \( \{ (\text{val}(A), \text{val}(B)) \} = h, \)
2. not only \((\text{val}(A), \text{val}(B)) \in h\)

On the other hand:

\[ \beta, h \models A \leftrightarrow B \ast \text{true} \iff (\text{val}(A), \text{val}(B)) \in h \]
Caution

\( \beta, h \models A \leftrightarrow B \) means that:

- \( \{ (\text{val}(A), \text{val}(B)) \} = h \),
- not only \( (\text{val}(A), \text{val}(B)) \in h \)

On the other hand:

\( \beta, h \models A \leftrightarrow B * \text{true} \iff (\text{val}(A), \text{val}(B)) \in h \)

Notation sometimes: \( A \leftrightarrow B :\iff A \leftrightarrow B * \text{true} \)
Some Valid Formulas

- $\text{emp} \iff \neg(\exists x, y. x \mapsto y \ast true)$
Some Valid Formulas

- $\text{emp} \iff \neg (\exists x, y. \ x \mapsto y \ast true)$

- $\varphi \ast \psi \iff \varphi \land \psi$
  
  if neither emp nor $\mapsto$ occur.
Some Valid Formulas

- \( \text{emp} \leftrightarrow \neg (\exists x, y. \ x \mapsto y \ast \text{true}) \)

- \( \varphi \ast \psi \leftrightarrow \varphi \land \psi \)
  
  if neither emp nor \( \mapsto \) occur.

- \( x \mapsto y \land x \mapsto z \rightarrow y = z \)
Some Valid Formulas

- emp $\leftrightarrow \neg(\exists x, y. x \mapsto y \ast true)$

- $\varphi \ast \psi \leftrightarrow \varphi \land \psi$
  if neither emp nor $\mapsto \ast$ occur.

- $x \mapsto y \land x \mapsto z \rightarrow y = z$

- $P \ast (Q \lor R) \leftrightarrow (P \ast Q) \lor (P \ast R)$
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula \( \varphi \) is satisfiable iff there are \( \beta \) and \( h \) such that \( \beta, h \models \varphi \).)

1. \( x \mapsto y \ast x \mapsto z \)
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula $\varphi$ is satisfiable iff are there $\beta$ and $h$ such that $\beta, h \models \varphi$.)

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2. $x \mapsto y \land x \mapsto z$
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1. $x \mapsto y \ast x \mapsto z$
2. $x \mapsto y \land x \mapsto z$
3. $(x \mapsto 0 \land y \mapsto 0) \rightarrow x = y$
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1. \( x \mapsto y \ast x \mapsto z \)
2. \( x \mapsto y \land x \mapsto z \)
3. \( (x \mapsto 0 \land y \mapsto 0) \rightarrow x = y \)
4. \( (x \mapsto 0 \ast y \mapsto 0) \rightarrow x = y \)
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2. $x \mapsto y \land x \mapsto z$
3. $(x \mapsto 0 \land y \mapsto 0) \rightarrow x = y$
4. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow x = y$
5. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg(x = y)$
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula $\varphi$ is satisfiable iff there are $\beta$ and $h$ such that $\beta, h \models \varphi$.)

1. $x \mapsto y \ast x \mapsto z$
2. $x \mapsto y \land x \mapsto z$
3. $(x \mapsto 0 \land y \mapsto 0) \rightarrow x = y$
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5. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg(x = y)$
6. $(x \mapsto a \land y \mapsto b) \rightarrow a = b$
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4. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow x = y$
5. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg(x = y)$
6. $(x \mapsto a \land y \mapsto b) \rightarrow a = b$
7. $\varphi \ast \text{emp} \rightarrow \varphi$
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula \( \varphi \) is satisfiable iff there \( \beta \) and \( h \) such that \( \beta, h \models \varphi \).)

1. \( x \mapsto y \ast x \mapsto z \)
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5. \( (x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg(x = y) \)
6. \( (x \mapsto a \land y \mapsto b) \rightarrow a = b \)
7. \( \varphi \ast \text{emp} \rightarrow \varphi \)
8. \( \varphi \ast \neg\varphi \)
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula $\varphi$ is satisfiable iff are there $\beta$ and $h$ such that $\beta, h \models \varphi$.)

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5. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg(x = y)$
6. $(x \mapsto a \land y \mapsto b) \rightarrow a = b$
7. $\varphi \ast \text{emp} \rightarrow \varphi$
8. $\varphi \ast \neg \varphi$

a. $\psi \ast \neg \psi$ for $\psi$ without $\mapsto$, emp
Quiz!

Are the following formulas valid/satisfiable/unsatisfiable?
(here: A formula $\varphi$ is satisfiable iff are there $\beta$ and $h$ such that $\beta, h \models \varphi$.)

1. $x \mapsto y \ast x \mapsto z$
2. $x \mapsto y \land x \mapsto z$
3. $(x \mapsto 0 \land y \mapsto 0) \rightarrow x = y$
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5. $(x \mapsto 0 \ast y \mapsto 0) \rightarrow \neg (x = y)$
6. $(x \mapsto a \land y \mapsto b) \rightarrow a = b$
7. $\varphi \ast \text{emp} \rightarrow \varphi$
8. $\varphi \ast \neg \varphi$

a. $\psi \ast \neg \psi$

for $\psi$ without $\mapsto$, emp

b. $x \mapsto y \ast \neg (x \mapsto y)$
The Magic Wand

Modus Ponens for classical logic

\[
\begin{align*}
A \land (A \rightarrow B) & \\
\hline
B
\end{align*}
\]
The Magic Wand

Modus Ponens for classical logic

\[ A \land (A \to B) \]

---

Corresponding rule for separating conjunction *? 

\[ A \land (A \to B) \]

---

Corresponding rule for separating conjunction *?
The Magic Wand

Modus Ponens for classical logic

\[
A \land (A \rightarrow B) \quad \Rightarrow \quad B
\]

Corresponding rule for separating conjunction ∗?

Modus Ponens for separation logic

\[
A \ast (A \ast B) \quad \Rightarrow \quad B
\]

The magic wand operator \( A \ast B \), aka separating implication:

\[\beta, h \models A \ast B\]

\[\iff\]

for all \( h', h^+ : \mathbb{N} \rightarrow \mathbb{N} \): If \( h^+ = h \cup h' \) and \( h' \models A \), then \( h^+ \models B \)
Separating Operators

\( \Diamond \models_{\text{SL}} f \ast g \quad \text{when there are } \Diamond \text{ and } \mathcal{D} \)

\( \text{such that } \Diamond = \mathcal{D}, \text{ as well as } \Diamond \models_{\text{SL}} f \text{ and } \mathcal{D} \models g. \)

\( \mathcal{D} \models_{\text{SL}} f \rightarrow g \quad \text{when any } \Diamond \text{ such that } \Diamond \models_{\text{SL}} f \text{ is also such that } \Diamond \models g. \)

Figure 1.5: Visual representation of the semantics of separation operators

Taken from:
Separation Logic: Expressiveness, Complexity, Temporal Extension
Rémi Brochenin, PhD Thesis. 2013
Programs and Separation Logic
Programming Language

\[
\text{statement} ::= \text{while formula do statement}
\]
\[
| \quad \text{if formula then statement else statement}
\]
\[
| \quad \text{statement} ; \text{statement}
\]
\[
| \quad \text{var := term}
\]
\[
| \quad \text{[term]} := \text{term}
\]
\[
| \quad \text{var := [term]}
\]
\[
\text{(later)} \quad | \quad \text{var := cons(term, ..., term)}
\]
\[
\text{(later)} \quad | \quad \text{dispose(var)}
\]

**Restriction:** formula are the arithmetic formulas that do not contain \(\implies\) or emp.
- Every state is a pair \((\beta, h)\) with \(\beta : \text{Var} \rightarrow \mathbb{N}\) and \(h : \mathbb{N} \rightarrow \mathbb{N}\)
- Kripke state transition the program semantics \(\rho(st) \in S \times S\) for any statement \(st\).
Program semantics (repetition from FODL)

Accessibility Relation for Programs

$\rho : \text{statement} \rightarrow S \times S$
Program semantics (repetition from FODL)

**Accessibility Relation for Programs**

\[ \rho : \text{statement} \rightarrow S \times S \]

\[ \rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2) \]
Program semantics (repetition from FODL)

Accessiblity Relation for Programs

\( \rho : \text{statement} \rightarrow S \times S \)

\[
\rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2)
\]

\[
\rho(\pi_1 ; \pi_2) = \rho(\pi_1) ; \rho(\pi_2) \quad \text{; is forward composition}
\]
Program semantics (repetition from FODL)

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\rho(\pi_1 ; \pi_2) = \rho(\pi_1) ; \rho(\pi_2) \quad ; \text{is forward composition}
\]

\[
= \{(s, t) \mid \text{ex. } u \in S \text{ with } (s, u) \in \rho(\pi_1), (u, t) \in \rho(\pi_2)\}
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\[
\rho(\pi^*) = \rho(\pi)^* \quad \text{* is refl. transitive closure}
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Program semantics (repetition from FODL)

Accessiblity Relation for Programs

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\rho(\pi^*) = \rho(\pi)^* \quad \text{* is refl. transitive closure}
\]

\[
= \{(s_o, s_n) \mid \text{ex. } n \geq 0 \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ f.a. } i < n\}
\]
Program semantics (repetition from FODL)

Accessibility Relation for Programs

\[ \rho : \text{statement} \rightarrow S \times S \]

\[ \rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2) \]

\[ \rho(\pi_1 ; \pi_2) = \rho(\pi_1) ; \rho(\pi_2) \quad \text{; is forward composition} \]

\[ = \{ (s, t) \mid \text{ex. } u \in S \text{ with } (s, u) \in \rho(\pi_1), (u, t) \in \rho(\pi_2) \} \]

\[ \rho(\pi^*) = \rho(\pi)^* \quad \text{* is refl. transitive closure} \]

\[ = \{ (s_0, s_n) \mid \text{ex. } n \geq 0 \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ f.a. } i < n \} \]

\[ \rho(?\varphi) = \{ (s, s) \mid s \models \varphi \} \]
**Accessiblity Relation for Programs**

\[ \rho : \text{statement} \rightarrow S \times S \]

\[ \rho(\pi_1 \cup \pi_2) = \rho(\pi_1) \cup \rho(\pi_2) \]

\[ \rho(\pi_1 ; \pi_2) = \rho(\pi_1) ; \rho(\pi_2) \quad ; \text{is forward composition} \]

\[ = \{ (s, t) \mid \text{ex. } u \in S \text{ with } (s, u) \in \rho(\pi_1), (u, t) \in \rho(\pi_2) \} \]

\[ \rho(\pi^*) = \rho(\pi)^* \quad * \text{ is refl. transitive closure} \]

\[ = \{ (s_o, s_n) \mid \text{ex. } n \geq 0 \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ f.a. } i < n \} \]

\[ \rho(\varphi) = \{ (s, s) \mid s \models \varphi \} \]

**Reminder: IF and WHILE**

\[ \text{if } \varphi \text{ then } \alpha \text{ else } \beta = (\varphi ; \alpha) \cup (\neg \varphi ; \beta) \]

\[ \text{while } \varphi \text{ do } \alpha = (\varphi ; \alpha)^* ; \neg \varphi \]
Program semantics (with heap)

Accessibility Relation for Programs

\[ \rho : \text{statement} \rightarrow S \times S \]
A state \( s \in S \) is a pair \((\beta, h)\) with \( \beta : \text{Var} \rightarrow \mathbb{N} \) and \( h : \mathbb{N} \rightarrow \mathbb{N} \)
Program semantics (with heap)

Accessibility Relation for Programs

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\[
((\beta, h), (\beta', h')) \in \rho(v := t) \iff \beta' = \beta[v/\text{val}_\beta(t)] \text{ and } h' = h
\]
Program semantics (with heap)

Accessiblity Relation for Programs

\( \rho : \text{statement} \rightarrow S \times S \)

A state \( s \in S \) is a pair \((\beta, h)\) with \( \beta : \text{Var} \rightarrow \mathbb{N} \) and \( h : \mathbb{N} \rightarrow \mathbb{N} \)

\[
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\]

\[
\left( \left( \beta, h \right), \left( \beta', h' \right) \right) \in \rho(v := [t]) \quad \iff \quad \text{val}_\beta(t) \in \text{dom } h \text{ and } h' = h \text{ and } \\
\beta' = \beta[v/h[\text{val}_\beta(t)]]
\]
Program semantics (with heap)

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\]

\[
((\beta, h), (\beta', h')) \in \rho(v := [t]) \iff \text{val}_\beta(t) \in \text{dom } h \text{ and } h' = h \text{ and } \\
\beta' = \beta[v/h[\text{val}_\beta(t)]]
\]

\[
((\beta, h), (\beta', h')) \in \rho([t] := u) \iff \text{val}_\beta(t) \in \text{dom } h \text{ and } \beta' = \beta \text{ and } \\
h' = h[\text{val}_\beta(t)/\text{val}_\beta(u)]
\]

(Remember: \( f[a/b](a) = b \) and \( f[a/b](x) = f(x) \) for \( x \neq a \))
Failing executions

Statement \( x := [10] \) must not be executed if \( 10 \not\in \text{dom } h \).

State \((\beta, \emptyset)\) has no successor state in \( \rho(x := [10]) \).

How to distinguish between failed test \(?\psi\) and memory violation?
Failing executions

Statement \( x := [10] \) must not be executed if \( 10 \not\in \text{dom} \, h \).

State \((\beta, \emptyset)\) has no successor state in \( \rho(x := [10]) \).

How to distinguish between failed test \(?\psi\) and memory violation?

Model unallowed heap access:

\[
\text{fail} : \text{statement} \rightarrow 2^S
\]

\( s \in \text{fail}(\pi) \) means: \( \pi \) started in \( s \) may cause memory violation.
Failing executions

Model unallowed heap access:

\[ \text{fail : statement } \rightarrow 2^S \]

\[ s \in \text{fail}(\pi) \text{ means: } \pi \text{ started in } s \text{ may cause memory violation} \]

\[
\text{fail}(x := t) = \\
\text{fail}(?\psi) = \emptyset
\]
Failing executions

Model unallowed heap access:

\[ \text{fail : statement} \to 2^S \]

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\[
\begin{align*}
\text{fail}(x := t) &= \\
\text{fail}(?\psi) &= \emptyset \\
\text{fail}(x := [t]) &= \\
\text{fail}([t] := u) &= \{ (\beta, h) \mid \text{val}_\beta(t) \notin \text{dom } h \}
\end{align*}
\]
Failing executions

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\text{fail}(\pi_1 ; \pi_2) &= \text{fail}(\pi_1) \cup (\rho(\pi_1) ; \text{fail}(\pi_2)) \\
\text{fail}(\pi_1 \cup \pi_2) &= \text{fail}(\pi_1) \cup \text{fail}(\pi_2)
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Failing executions

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\text{fail}(\pi_1;\pi_2) &= \text{fail}(\pi_1) \cup (\rho(\pi_1);\text{fail}(\pi_2)) \\
\text{fail}(\pi_1 \cup \pi_2) &= \text{fail}(\pi_1) \cup \text{fail}(\pi_2) \\
\text{fail}(\pi^*) &= \rho(\pi^*) ; \text{fail}(\pi)
\end{align*}
\]
Failing executions

Model unallowed heap access:

\(fail : \text{statement} \rightarrow 2^S\)

\(s \in fail(\pi)\) means: \(\pi\) started in \(s\) may cause memory violation

\[
\begin{align*}
fail(x := t) & = \\
fail(?\psi) & = \emptyset \\
fail(x := [t]) & = \\
fail([t] := u) & = \{(\beta, h) \mid \text{val}_\beta(t) \notin \text{dom } h\} \\
fail(\pi_1 ; \pi_2) & = fail(\pi_1) \cup (\rho(\pi_1) ; fail(\pi_2)) \\
fail(\pi_1 \cup \pi_2) & = fail(\pi_1) \cup fail(\pi_2) \\
fail(\pi^*) & = \rho(\pi^*) ; fail(\pi)
\end{align*}
\]

with \(A ; B = \{x \mid \text{ex } y \text{ with } (x, y) \in A \text{ and } y \in B\}\)
Fail-aware modality

Remember:

\[ s \models [\pi] \varphi \quad \text{iff} \quad s' \models \varphi \quad \text{for all} \quad (s, s') \in \rho(\pi). \]

Problem:

emp \rightarrow [[5] := 42] false \quad \text{is a valid formula.}

New modality \([\cdot]\):

\[ s \models [\pi] \varphi \quad \text{iff} \quad s' \models \varphi \quad \text{for all} \quad (s, s') \in \rho(\pi) \quad \text{and} \quad s \notin \text{fail}(\pi) \]

Now:

\[ \text{emp} \rightarrow [[5] := 42] \psi \quad \text{is not valid for any} \quad \psi \]
Dynamic Separation Logic

Valid formulas:

- \( x \mapsto 5 \rightarrow [[v := [x] ; [x] := v + 1]x \mapsto 6 \)

- \( (\exists y. x \mapsto y) \rightarrow [[x] := 7]x \mapsto 7 \)

- \( x \mapsto 5 \ast y \mapsto 6 \rightarrow [[x] := 7](x \mapsto 7 \ast y \mapsto 6) \)
### Hoare Calculus

Separation Logic originally formulated as rules for a *Hoare* calculus.
A Calculus for Separation Logic

Hoare Calculus

Separation Logic originally formulated as rules for a Hoare calculus.

Hoare Calculus (1969, Hoare and Floyd)

Operates on Hoare Triples: \( \{ P \} \pi \{ Q \} \)

A Hoare triple is valid if program \( \pi \) started in a state that satisfies precondition \( P \) terminates in a state which satisfies postcondition \( Q \) (it it terminates).
Hoare Calculus

Separation Logic originally formulated as rules for a *Hoare* calculus.

Hoare Calculus (1969, Hoare and Floyd)

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Semantically the same as \( P \rightarrow [\pi]Q \).
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Separation Logic originally formulated as rules for a *Hoare* calculus.

**Hoare Calculus (1969, Hoare and Floyd)**

Operates on **Hoare Triples**: \( \{P\} \pi \{Q\} \)

A Hoare triple is valid if program \( \pi \) started in a state that satisfies precondition \( P \) terminates in a state which satisfies postcondition \( Q \) (it it terminates).

Semantically the same as \( P \rightarrow [\pi] Q \).

We present the calculus using dynamic logic notation.
Reminder: Hoare Calculus (in DL notation)

\[ P[x\leftarrow E] \rightarrow [x := E]P \]  
[\(x\leftarrow E\)] is substitution

\[ \begin{align*} 
P \rightarrow [\pi_1]Q & \quad Q \rightarrow [\pi_2]R \\
P \rightarrow [\pi_1 ; \pi_2]R & \end{align*} \]

\[ \begin{align*} 
P' \rightarrow P & \quad P \rightarrow [\pi]Q \\
Q \rightarrow Q' & \end{align*} \]

\[ \begin{align*} 
P \land C \rightarrow [\pi_1]Q & \quad P \land \neg C \rightarrow [\pi_2]Q \\
P \rightarrow [\text{if } C \text{ then } \pi_1 \text{ else } \pi_2]Q & \end{align*} \]

\[ \begin{align*} 
P \land C \rightarrow [\pi]P & \end{align*} \]

\[ \begin{align*} 
P \rightarrow [\text{while } C \text{ do } \pi](P \land \neg C) & \end{align*} \]

\[ \begin{align*} 
P \rightarrow [\pi]Q & \quad (\exists x. P) \rightarrow [\pi](\exists x. Q) \\
\text{if } x \notin \text{Free}(\pi) & \end{align*} \]
Axioms:

\[ x = m \land \text{emp} \rightarrow [x := E] x = E[x \leftarrow m] \land \text{emp} \]

\[ x = m \land E \rightarrow n \rightarrow [x := [E]](x = n \land E[x \leftarrow m] \rightarrow n) \]

\[ (E \rightarrow n) \rightarrow [[E] := F] E \rightarrow F \]

Heap location must be accessible

Recall: \( s \models [\pi] \varphi \) iff \( s' \models \varphi \) for all \((s, s') \in \rho(\pi)\) and \(s \notin \text{fail}(\pi)\). All accessed heap locations (read or write) must be in domain. Therefore: Precondition must ensure that.
Separation Logic Rules for Assignments

**Axioms:**

\[ x = m \land \text{emp} \rightarrow [x := E]x = E[x\leftarrow m] \land \text{emp} \]

\[ x = m \land E \hookrightarrow n \rightarrow [x := [E]](x = n \land E[x\leftarrow m] \hookrightarrow n) \]

\[ (\exists n. E \hookrightarrow n) \rightarrow [[E] := F] E \hookrightarrow F \]

**Heap location must be accessible**

Recall: \( s \models [\pi]\varphi \) iff \( s' \models \varphi \) for all \((s, s') \in \rho(\pi)\) and \( s \not\in \text{fail}(\pi) \).

All accessed heap locations (read or write) must be in domain.

Therefore: Precondition must ensure that.
The Frame Rule

\[
\begin{align*}
P & \quad \rightarrow \quad [\pi] \quad Q \\
\quad P \ast R & \quad \rightarrow \quad [\pi](Q \ast R)
\end{align*}
\]

\[\text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset\]
The Frame Rule

**THIS IS THE KEY POINT ABOUT SEPARATION LOGIC**

\[
P \quad \rightarrow \quad [\pi] \quad Q
\]

\[
P \ast R \quad \rightarrow \quad [\pi](Q \ast R)
\]

\[
\text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset
\]

**Separation in Proofs**

Proof \( P \rightarrow [\pi] Q \) using in \( P, Q \) the memory \( \pi \) refers to.

Get for free: Nothing besides these memory locations has changed.
Remember: The Framing Problem

Example in Java

```java
//@ requires acc1 != acc2;
//@ ensures \result == 100;
int f(Account acc1, Account acc2) {
    acc1.setBalance(100);
    acc2.setBalance(200);
    return acc1.getBalance();
}
```

Rule for setBalance:

\[ A \mapsto x \rightarrow [A.setBalance(y)] A \mapsto y \]
Example in Java

```java
//@ requires acc1 != acc2;
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int f(Account acc1, Account acc2) {
    acc1.setBalance(100);
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}
```

Rule for setBalance:

\[ A \mapsto x \rightarrow \llbracket A.setBalance(y) \rrbracket A \mapsto y \]

Use Frame Rule:

\[ acc2 \mapsto x \rightarrow \ldots \]

\[ \ldots \llbracket acc2.setBalance(200); \rrbracket acc2 \mapsto 200 \]
Remember: The Framing Problem

Example in Java

```java
//@ requires acc1 != acc2;
//@ ensures \result == 100;
int f(Account acc1, Account acc2) {
    acc1.setBalance(100);
    acc2.setBalance(200);
    return acc1.getBalance();
}
```

Rule for `setBalance`:

\[
A \mapsto x \rightarrow [A.setBalance(y)]A \mapsto y
\]

Use Frame Rule:

\[
acc2 \mapsto x * \quad acc1 \mapsto 100 \quad \rightarrow \quad \ldots
\]
\[
\ldots [acc2.setBalance(200); ]acc2 \mapsto 200 * \quad acc1 \mapsto 100
\]
(∃v. X ↦ v ∗ Y ↦ v) → [X := [X] ; Y := [Y]] X = Y
Soundness of Frame Rule

\[ \begin{align*}
    P & \rightarrow [\pi] Q \\
    P \ast R & \rightarrow [\pi](Q \ast R)
\end{align*} \]

or equivalently

\[ [\pi] Q \ast R \rightarrow [\pi](Q \ast R) \]

if \( \text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset \)
Soundness of Frame Rule

\[
\frac{P \rightarrow \lbrack \pi \rbrack Q}{P \ast R \rightarrow \lbrack \pi \rbrack (Q \ast R)} \quad \text{or equivalently} \quad \models (\lbrack \pi \rbrack Q) \ast R \rightarrow \lbrack \pi \rbrack (Q \ast R)
\]

if \( \text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset \)

\[\implies\]

Instantiate left rule with \( P := \lbrack \pi \rbrack Q \).

Premiss: trivially true, conclusion: desired implication.
Soundness of Frame Rule

\[
\begin{align*}
P & \rightarrow \llbracket \pi \rrbracket Q \\
\frac{P \ast R}{P \ast R \rightarrow \llbracket \pi \rrbracket (Q \ast R)}
\end{align*}
\]

or equivalently

\[
\models (\llbracket \pi \rrbracket Q) \ast R \rightarrow \llbracket \pi \rrbracket (Q \ast R)
\]

if \( \text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset \)

\[\implies\]

Instantiate left rule with \( P := \llbracket \pi \rrbracket Q \).

Premiss: trivially true, conclusion: desired implication.

\[\iff\]

Let \( \beta, h \models P \ast R \), i.e., \( \beta, h_1 \models P \) and \( \beta, h_2 \models R \) with \( h = h_1 \cup h_2 \).

By premiss: \( \beta, h_1 \models \llbracket \pi \rrbracket Q \) and \( \beta, h \models (\llbracket \pi \rrbracket Q) \ast R \)

Right rule gives: \( \beta, h \models \llbracket \pi \rrbracket (Q \ast R) \)
Soundness of Frame Rule

\[
\frac{P \rightarrow [\pi] Q}{P * R \rightarrow [\pi](Q * R)} \quad \text{or equivalently} \quad \models ([\pi] Q) * R \rightarrow [\pi](Q * R)
\]

if \(\text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset\)

\[\equiv\]

Instantiate left rule with \(P := [\pi] Q\).
Premiss: trivially true, conclusion: desired implication.

\[\leftarrow\]

Let \(\beta, h \models P * R\), i.e., \(\beta, h_1 \models P\) and \(\beta, h_2 \models R\) with \(h = h_1 \cup h_2\). By premiss: \(\beta, h_1 \models [\pi] Q\) and \(\beta, h \models ([\pi] Q) * R\)
Right rule gives: \(\beta, h \models [\pi](Q * R)\)
Lemma

Let $h_1, h'_1, h_2, h'_2 : \mathbb{N} \to \mathbb{N}$ be heaplets, $\text{dom } h_1 \cap \text{dom } h_2 = \emptyset$

$\beta, \beta' : \text{Var} \to \mathbb{N}$ be variable assignments. Then:

$\beta, h_1 \xrightarrow{\pi} \beta', h'_1 \implies (\beta, h_1 \cup h_2 \xrightarrow{\pi} \beta', h'_1 \cup h'_2 \iff h_2 = h'_2)$

$s \xrightarrow{\pi} s'$ means $(s, s') \in \rho(\pi)$
Lemma

Let \( h_1, h'_1, h_2, h'_2 : \mathbb{N} \rightarrow \mathbb{N} \) be heaplets, \( \text{dom} \ h_1 \cap \text{dom} \ h_2 = \emptyset \).

\( \beta, \beta' : \text{Var} \rightarrow \mathbb{N} \) be variable assignments. Then:

\[
\beta, h_1 \xrightarrow{\pi} \beta', h'_1 \quad \implies \quad (\beta, h_1 \cup h_2 \xrightarrow{\pi} \beta', h'_1 \cup h'_2 \iff h_2 = h'_2)
\]

By structural induction:

- variable assignment \( v := t \) (heap irrelevant)
- heap store \( [t_1] := t_2 \) (\( \text{val}(t) \notin \text{dom} \ h_1 \))
- heap load \( v := [t] \) (\( \text{val}(t) \notin \text{dom} \ h_1 \))
- first-order test \( ?\varphi \) (heap irrelevant)
- \( \pi_1 \cup \pi_2, \pi_1 ; \pi_2, \pi^* \) (appeal to ind. hyp)

\[ s \xrightarrow{\pi} s' \text{ means } (s, s') \in \rho(\pi) \]
Soundness of Frame Rule

\[ \models ([\pi] Q) \ast R \rightarrow [\pi](Q \ast R) \quad \text{if } \text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset \quad (*) \]

Let \( \beta, h \models ([\pi] Q) \ast R \), i.e., \( \beta, h_1 \models [\pi] Q \) and \( \beta, h_2 \models R \), \( h = h_1 \cup h_2 \).

\[ \beta, h_1 \models ([\pi] Q) \quad \pi \quad \beta', h_1 Q \models Q \]

\[ \beta, h \models ([\pi] Q) \ast R \]

\[ \beta, h_2 \models R \quad (\ast) \quad \beta', h_2 \models R \]

\[ \beta', h_1 Q \cup h_2 \models Q \ast R \]

**Lemma, lang. is deterministic** \( \rightarrow \beta, h \models [\pi](Q \ast R) \)
Memory Allocation and Deallocation

Syntax: Two statements

\[
\text{var} := \text{cons}(\text{term, ...}, \text{term}) \quad \text{and} \quad \text{dispose}(\text{var})
\]
Memory Allocation and Deallocation

Syntax: Two statements

\[
\text{var} := \text{cons}(\text{term}, \ldots, \text{term}) \quad \text{and} \quad \text{dispose}(\text{var})
\]

Semantics: \( \rho \) and \( \text{fail} \)

\[
((\beta, h), (\beta', h')) \in \rho(v := \text{cons}(t)) \quad \text{iff} \quad \\
\beta' = \beta[v/\text{loc}] \quad \text{and} \quad h' = h \cup \{(\text{loc}, \text{val}_\beta(t))\} \quad \text{and} \quad \text{loc} \notin \text{dom} \, h
\]

\[
\text{fail}(v := \text{cons}(t)) = \emptyset
\]

\text{cons} allocates \( n \) consecutive unused memory locations, stores the argument values there and returns the first memory location.

(See literature for general \( n \)-ary version)
Memory Allocation and Deallocation

Syntax: Two statements

\[ \text{var} := \text{cons}(\text{term}, ..., \text{term}) \quad \text{and} \quad \text{dispose}(\text{var}) \]

Semantics: $\rho$ and $\text{fail}$

\[
((\beta, h), (\beta', h')) \in \rho(\text{dispose}(v))
\]

iff

\[
\beta' = \beta \quad \text{and} \quad \beta(v) \in \text{dom } h \quad \text{and} \quad h' = h \setminus \{ (\beta(v), h(\beta(v))) \}
\]

\[
\text{fail}(\text{dispose}(v)) = \{ (\beta, h) \mid \beta(v) \notin \text{dom } h \}
\]

dispose deallocates the allocated memory location $v$; fails if an unallocated location is disposed.
Soundness of Frame Rule

\[
\frac{([\pi] Q) * R} {\begin{array}{c}
\begin{cases}

([\pi] (Q * R)) \text{ if } \text{Modifies}(\pi) \cap \text{Free}(R) = \emptyset
\end{cases}
\end{array}}
\]

Proof by structural induction over \( \pi \).

see Reynolds p.77ff
Decidability of Separation Logic

Decidable

Some restricted logics from Separation Logic are decidable.

1. Restricted arithmetic
2. No magic wand $\rightarrow^*$

They can be reduced to Monadic Second Order Logic over $\mathbb{N}$. Equivalent to word emptiness of Büchi Automata.

The separating implication $\rightarrow^*$ makes undecidable.

Relatively complete

The calculus for Separation Logic is relatively complete. Every correct program can be proved using an oracle for $\mathbb{N}$. 
Application of Separation Logic
Abstraction Predicates

Use predicate symbols to abstract away from data structures

Example: Lists
Abstraction Predicates

Use predicate symbols to abstract away from data structures

Example: Lists

\[ \text{list}(x, \langle 17, 21, 9 \rangle) \leftrightarrow (x \mapsto 17) \ast (x + 1 \mapsto v) \ast (v \mapsto 21) \ast \ldots \ast (v + 1 \mapsto w) \ast (w \mapsto 9) \ast (w + 1 \mapsto 0) \]
Abstraction Predicates

Use predicate symbols to abstract away from data structures

**Example:** Lists

\[ \text{list}(x, \langle 17, 21, 9 \rangle) \leftrightarrow (x \mapsto 17) \ast (x+1 \mapsto v) \ast (v \mapsto 21) \ast \ldots \]
\[ \ldots \ast (v + 1 \mapsto w) \ast (w \mapsto 9) \ast (w + 1 \mapsto 0) \]

**General:**

Recursive predicate \textit{list}:

\[ \forall x, v_1, \bar{v}. \text{list}(x, \langle v_1, \bar{v} \rangle) \leftrightarrow \exists n. ((x \mapsto v_1) \ast (x+1 \mapsto n) \ast \text{list}(n, \bar{v})) \]
Verifast → Demo! (Bart Jacobs et al., U Leuven)
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Infer (Peter O’Hearn et al., Facebook)
http://fbinfer.com/
Program Verification Using Separation Logic

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SpaceInvader, YNot, HOLFoot, . . . , . . .
Discussion

Advantages of Separation Logic

+ Functional and frame specification combined – no extra consideration needed
+ Frame rule!
+ Abstraction Predicates are nice way of abstraction

Disadvantages of Separation Logic

− Functional and frame specification combined – no separation of concerns!
− All data must be hierarchically structured
− Complicated semantics of Sep Logic (c.f. −∗)