LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Theoretical Computer Science and Theorem Proving (TCSTP)
Institute for Informatics (IFI)

Proof Composition and Refinement of Voting Systems Using Isabelle/HOL

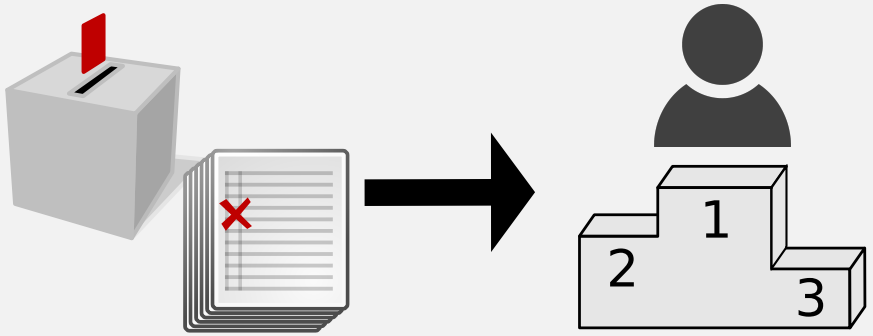
Michael Kirsten

Guest Lecture in Theorem Prover Lab at KIT, Karlsruhe

December 17, 2025

Structure of Voting Systems

Voting System



voting process

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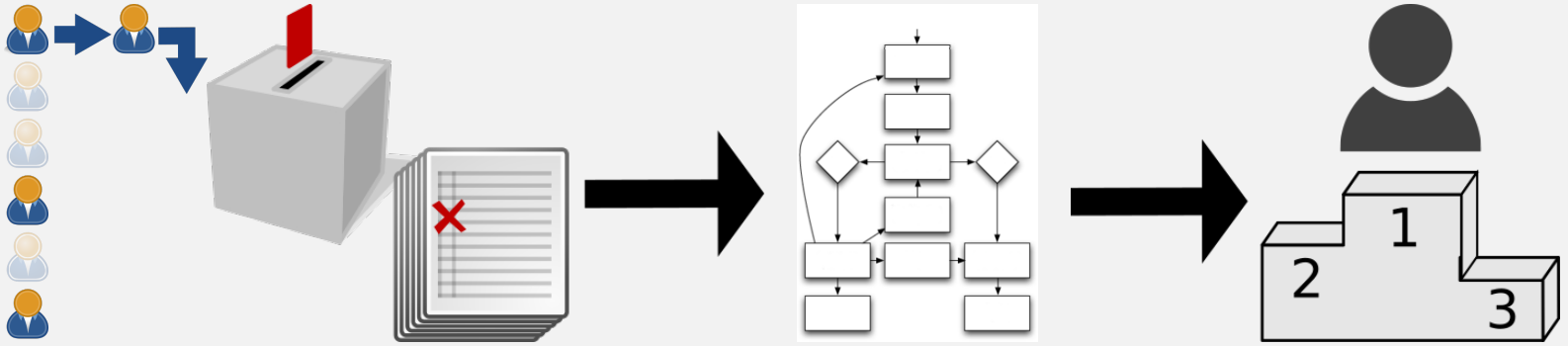

1. management 2. casting 3. tallying

How to tally the votes?

Vote Distribution

35 %	30 %	20 %	15 %
			

Who wins the election?

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Free and **equal** elections should ensure that . . .

- voters express their true will without distortion, and
- every vote counts the same.

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Example Criterion Against Strategies: Condorcet

Definition (Condorcet Criterion)

We ought to take the Condorcet winner as sole winner if it exists.

- *a beats b* if and only if more than half the voters prefer *a* to *b*.
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Problem

Sometimes there is no Condorcet winner.

$\Rightarrow$ *Condorcet paradox*

How can we ensure such properties?

Research Approach

- Proven construction of tallying algorithms based on *basic* modules
- Algorithm becomes comprehensible by such a construction
- Proof by application of proven *construction rules* $\Rightarrow$ **Composition of rules is explanation**
- Construction of tallying algorithms with desired properties (and explanation) can be automated

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- $\circlearrowleft_t$ repeats module until only one alternative remains $\Rightarrow$ **Condorcet winner wins if it exists**

Formal Robust-by-Construction Framework

Formal Methods

- Theorem prover for human-readable and machine-checked proofs
- Translation of mathematical functions into verified Scala/Haskell programs

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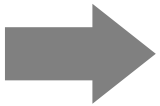
```
theorem condorcet_consistent_seq1:
  assumes condorcet_m: "condorcet_consistent m" and
          module_n:    "electoral_module n"
  shows "condorcet_consistent (m ▷ n)"
```



```
condorcetconsistent(seqcomp(X,Y)):-
    condorcetconsistent(X), module(Y).
```

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```
?- condorcetconsistent(X), electing(X), monotone(X).
  X = seqcomp(m_cond, m_borda) ;
  X = seqcomp(m_cond, m_plurality) ;
  X = seqcomp(m_cond, seqcomp(m_cond, m_borda)) .
```

Example: Sequential Majority Comparison

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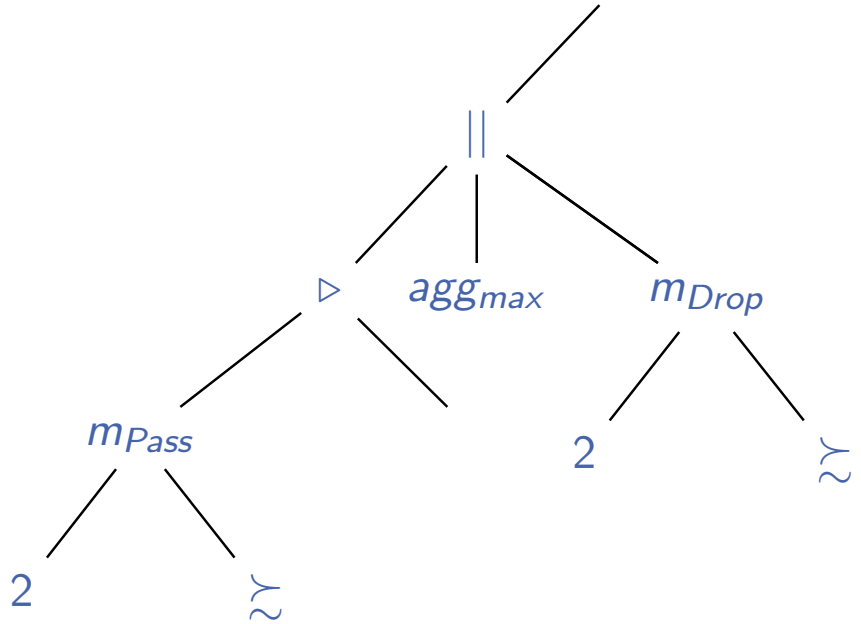
$((m_{Pass}(2) \triangleright \quad) \,||_{agg_{max}}\, m_{Drop}(2))$

- Pick **two** candidates.



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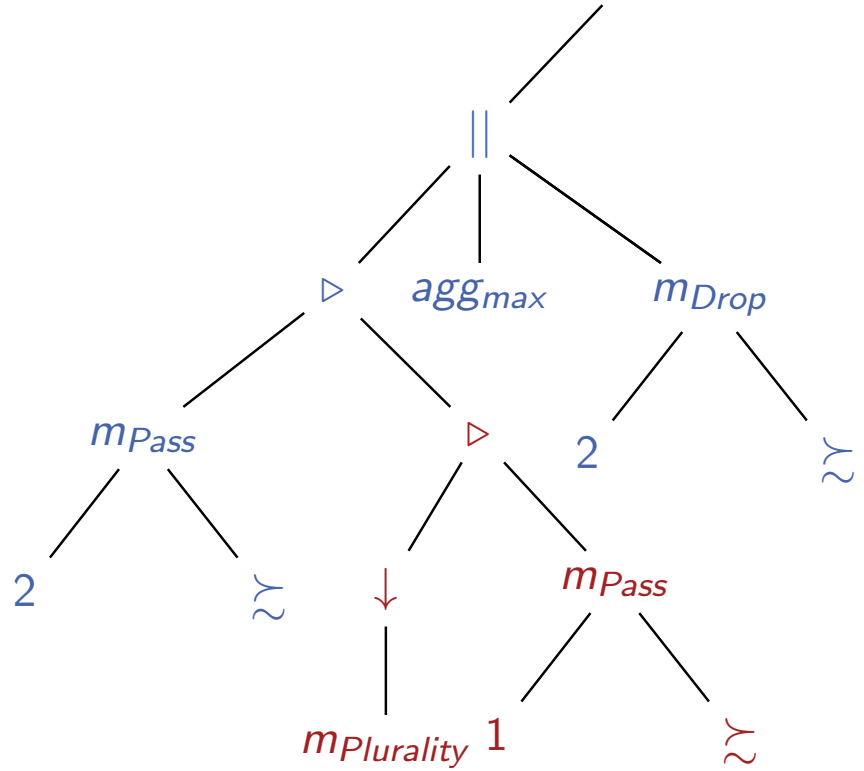

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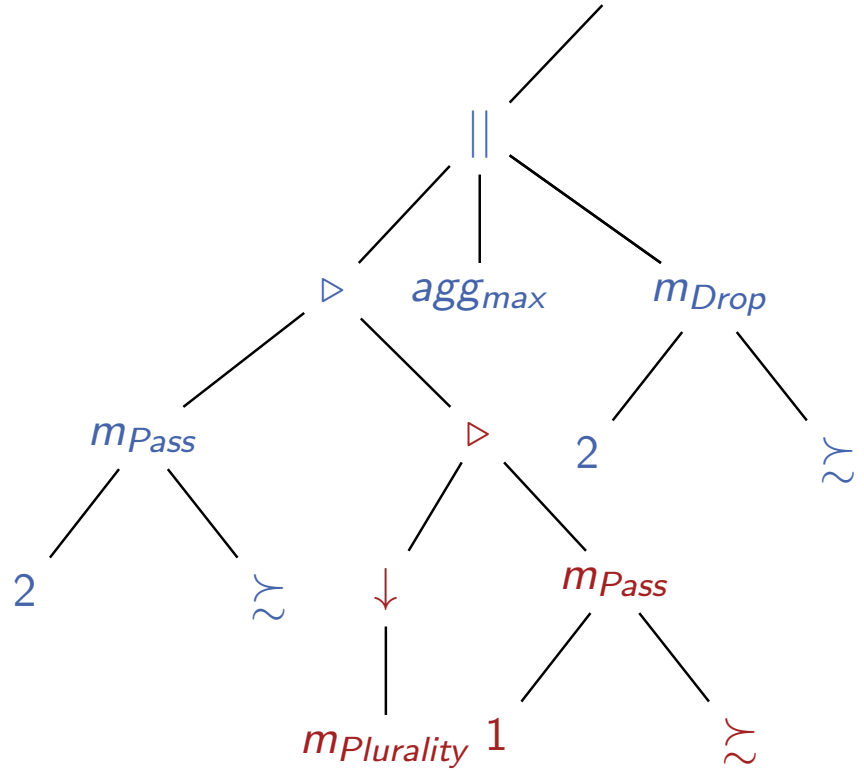

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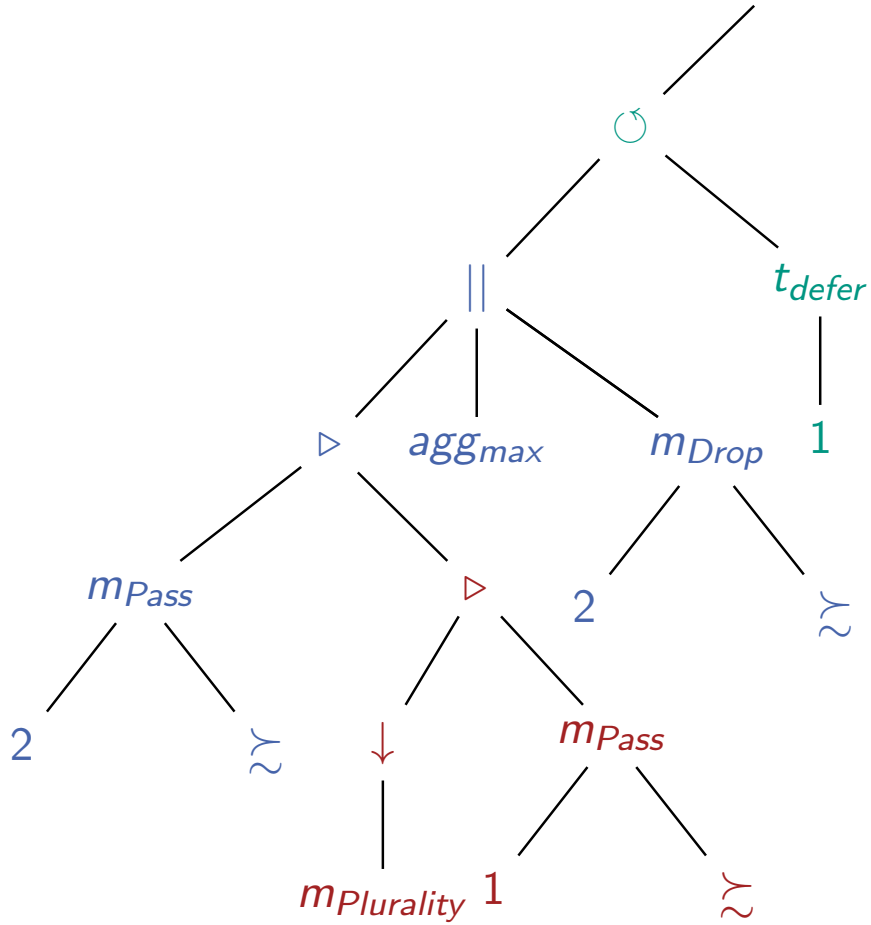
  - **Repeat**.



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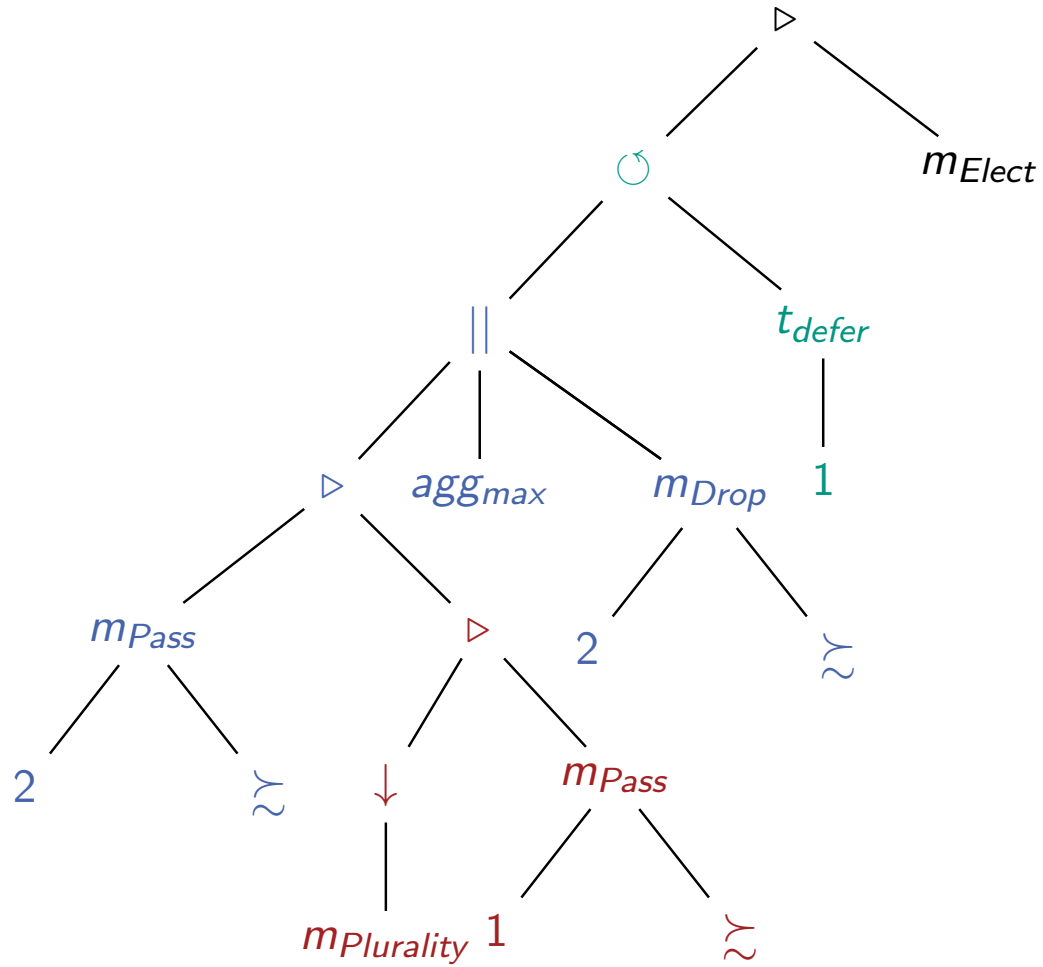

Proof Composition for Reliable Voting Rules

Proof of Monotonicity for Sequential Majority Comparison

$m_{elim} \circlearrowleft_{|d|=1} \triangleright m_{elect}$ is monotone

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$$\frac{m_{elect} \text{ is electing} \quad m_{elim}\circlearrowleft_{|d|=1} \text{ is non-electing} \quad m_{elim}\circlearrowleft_{|d|=1} \text{ defers 1 alternative} \quad m_{elim}\circlearrowleft_{|d|=1} \text{ is defer-lift-invariant}}{m_{elim}\circlearrowleft_{|d|=1} \triangleright m_{elect} \text{ is monotone}}$$

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All proven within Isabelle/HOL:

- 4 basic electoral modules
- 4 compositional structures
- 19 composition rules (10 reusable lemmas)
- 13 module properties

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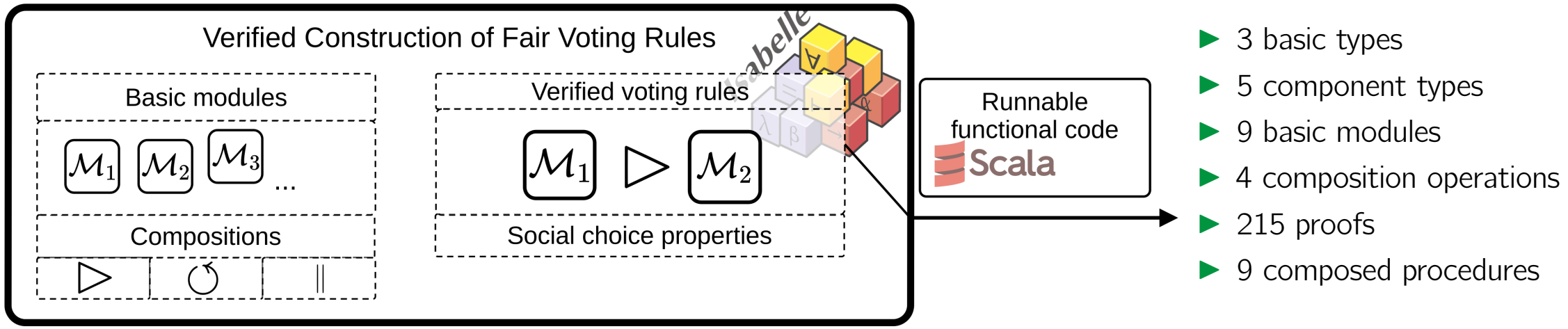
Framework for Correctness-by-Construction of Voting Rules



- 3 basic types
- 5 component types
- 9 basic modules
- 4 composition operations
- 215 proofs
- 9 composed procedures

Now Some Abstraction: The Elimination Set

Evaluation Function

- Idea: Assign an alternative's value by setting them into relation to the others
- Example: Minimax Score $m(a, A, p)$, the worst that alternative a does against any other alternative

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Elimination Set

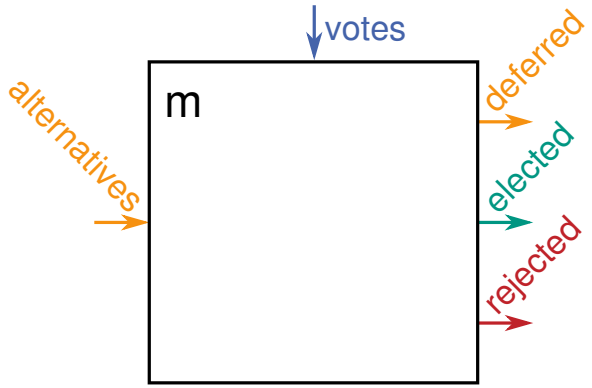
Eliminate each alternative whose value is in relation r with threshold value t.

$$E = \{a \in A \ : \ (e(a, A, p), t) \in r\} \text{ with}$$

- Eligible alternatives A, profile p
- Evaluation function e
- Threshold value $t \in \mathbb{N}$
- Binary relation r on $\mathbb{N}$

Making it a Module: The Elimination Module

Basic Module Type Inspired by **Grilli di Cortona et al. (1999)**



- Segmentation of outcome into set triple for further processing
- Common module type for general composition operations

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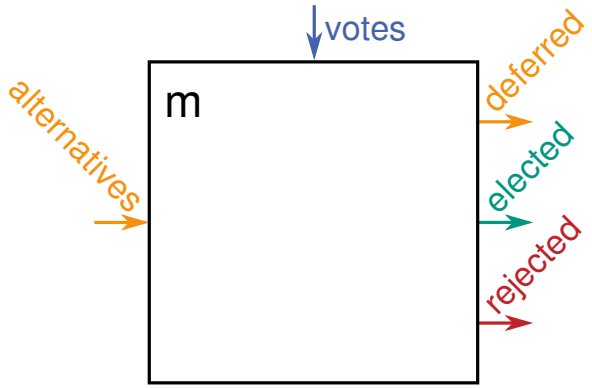

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Elimination Module

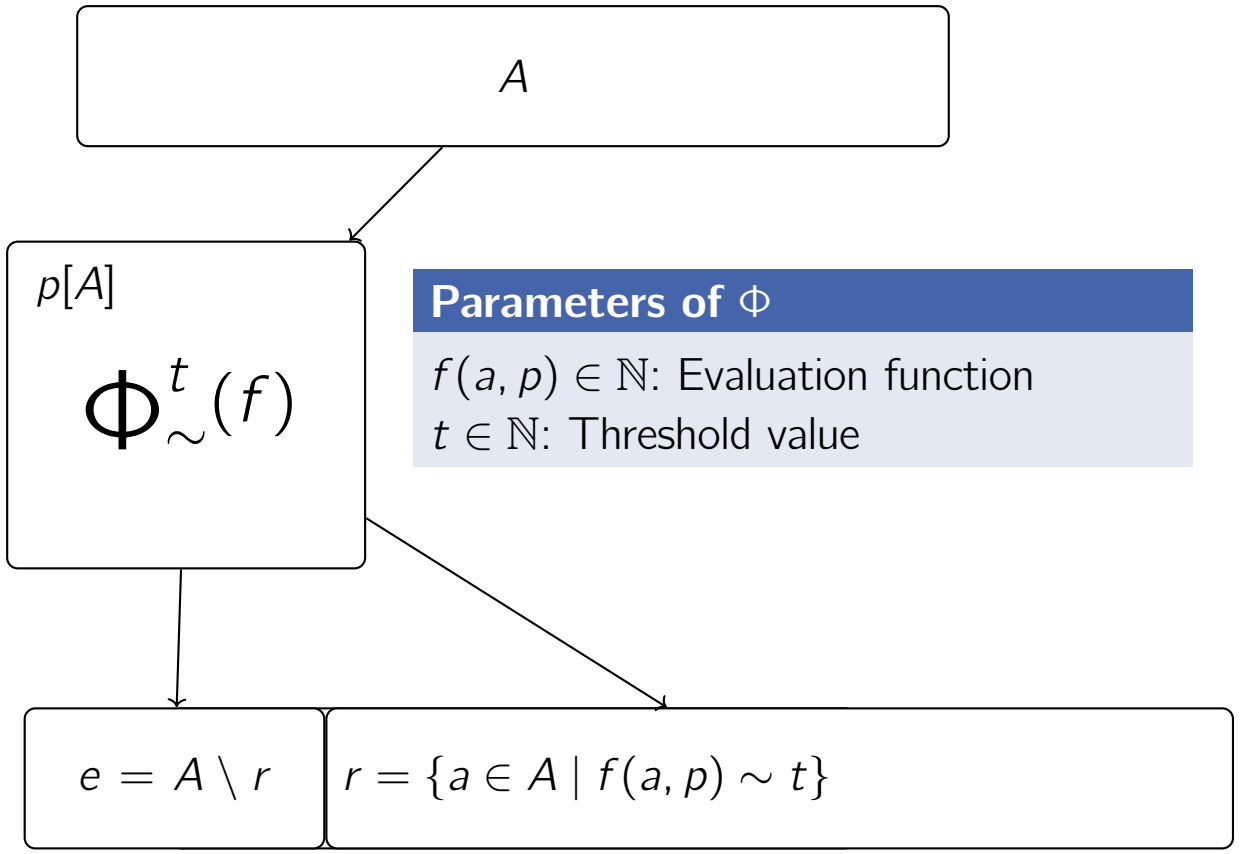
Reject elimination set if and only if we would not reject all alternatives.

$$m(A, p) := \begin{cases} (\varnothing, E, A \setminus E) & \text{if } E \subsetneq A \\ (\varnothing, \varnothing, A) & \text{else} \end{cases}$$

Special Cases

- LESS Eliminator: $a \in E \iff a$'s value $<$ threshold value
- MAX Eliminator: $a \in E \iff a$'s value $<$ maximum value of all alternatives

Turning the Elimination Module Into a Program



Parameters of Φ

$f(a,p) \in \mathbb{N}$: Evaluation function
$t \in \mathbb{N}$: Threshold value

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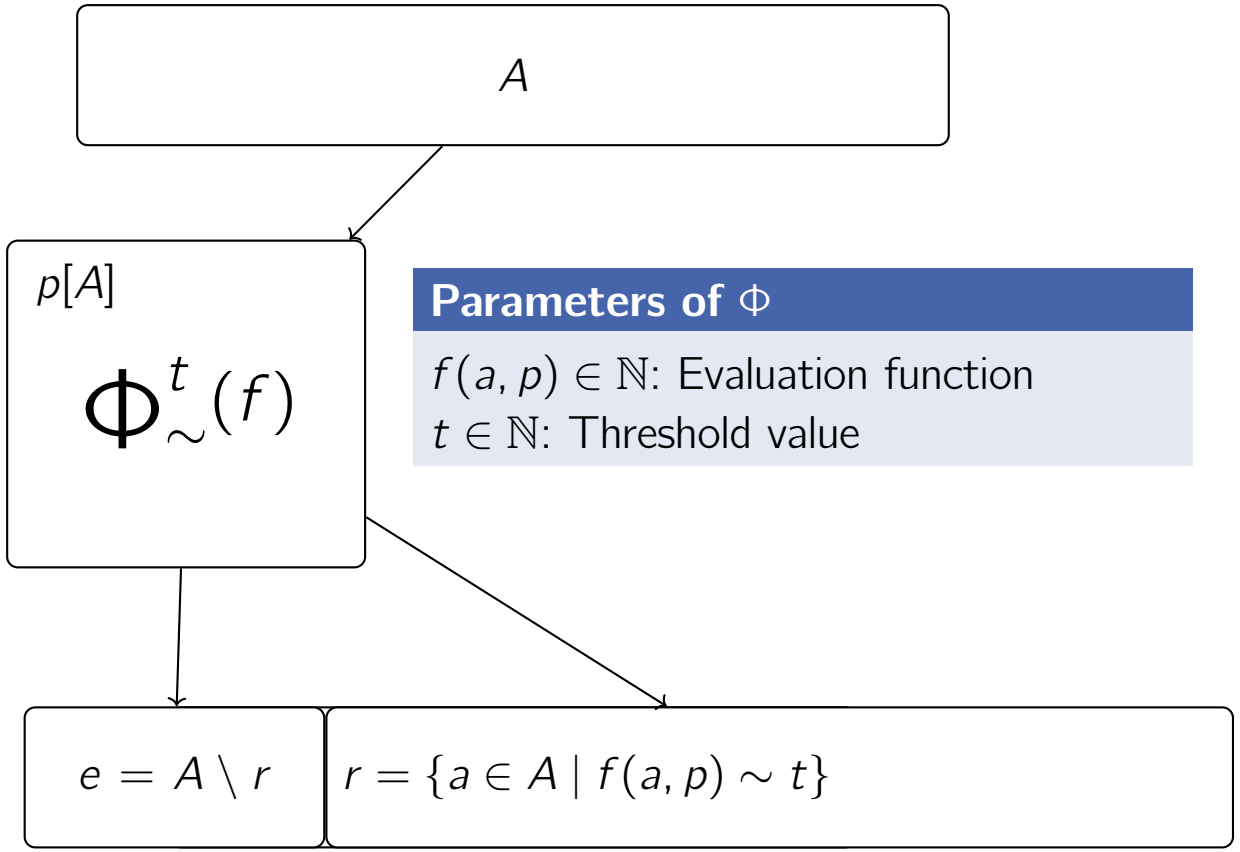
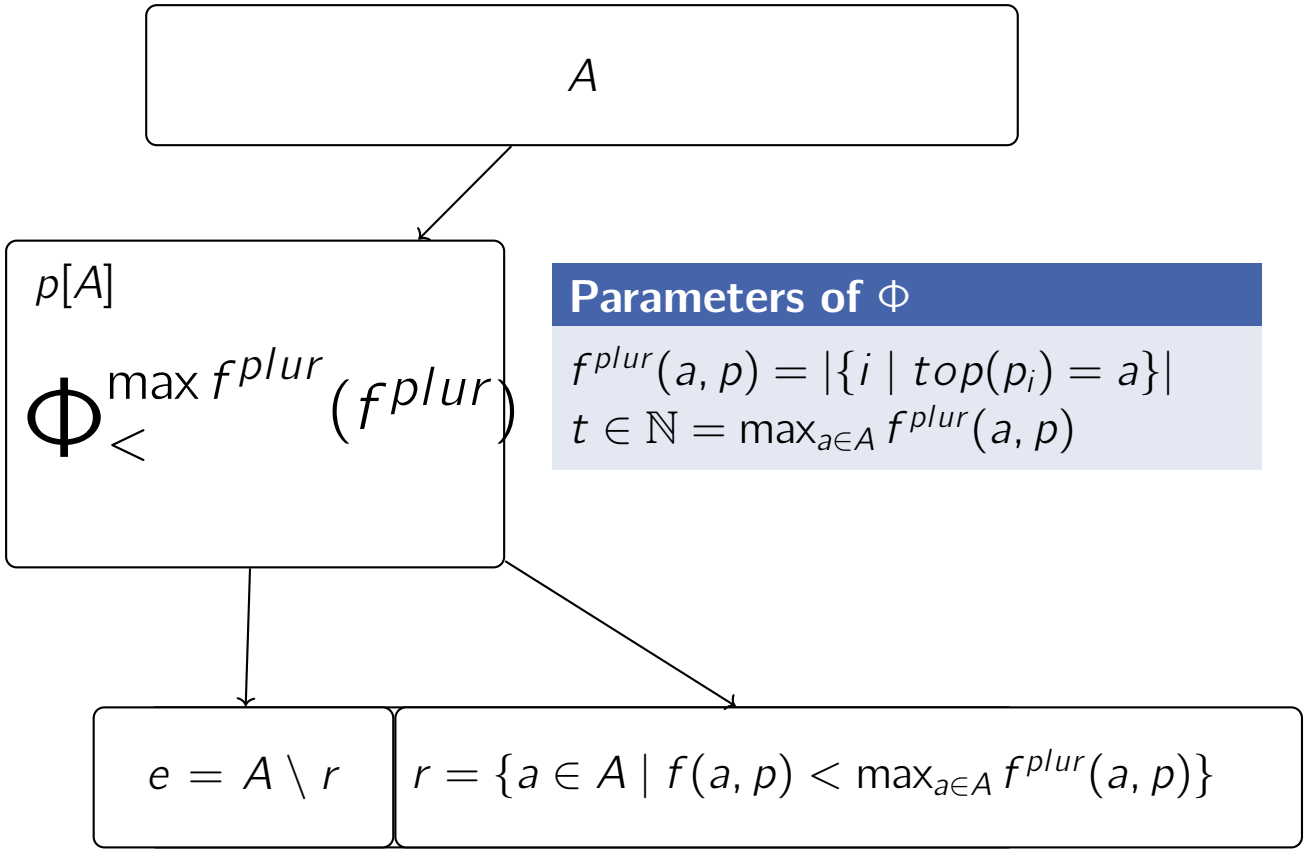


foreach $a \in A$ **do**
  if $f(a,p) \sim t$ **then** $r \leftarrow r \cup \{a\}$;
  else $e \leftarrow e \cup \{a\}$;
return (e, r);

Refined Elimination Module for the Plurality Rule



Parameters of Φ

$f^{plur}(a,p) = |\{i \mid top(p_i) = a\}|$

$t \in \mathbb{N} = \max_{a \in A} f^{plur}(a,p)$

$t \leftarrow \max_{a \in A} f^{plur}(a,p)$;
foreach $a \in A$ **do**
  if $f^{plur}(a,p) < t$ **then** $r \leftarrow r \cup \{a\}$;
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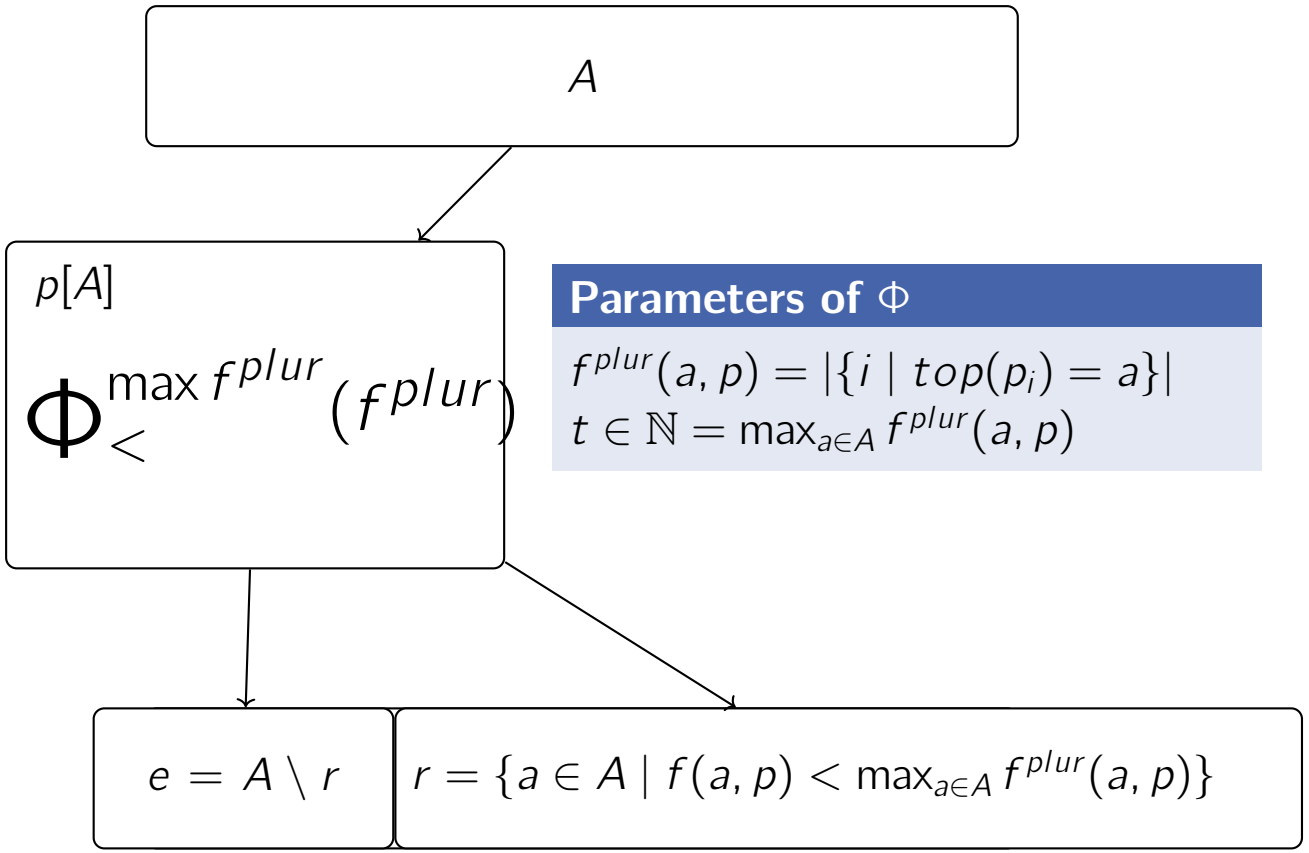

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$map : \tau_{alt} \to \mathbb{N}$;
for $a \in A$ **do** $map[a] \leftarrow f^{plur}(a,p)$;

assert $\forall a \in A\ map[a] = f^{plur}(a,p)$
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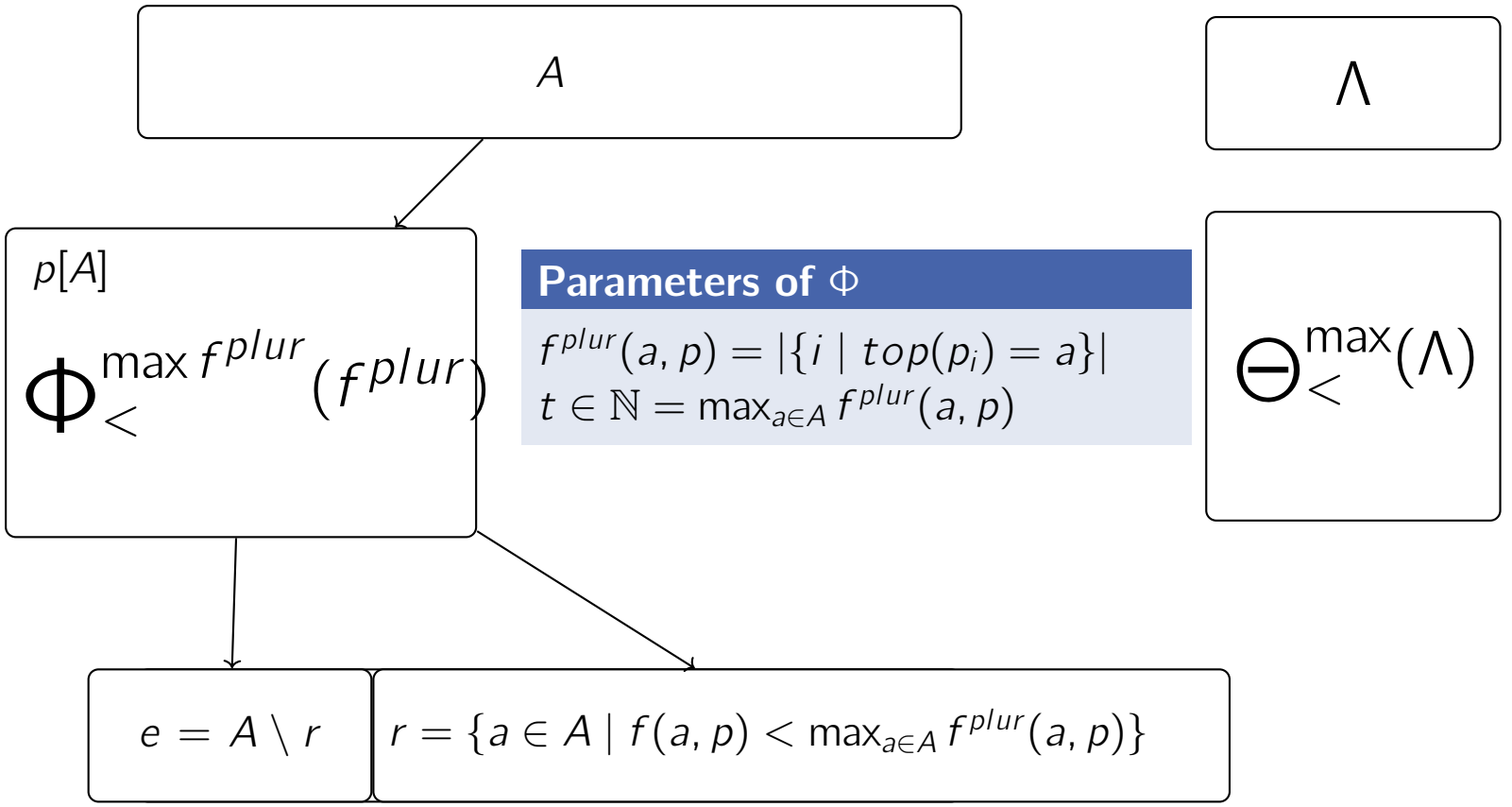

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Efficient Stepwise-Refined Programs Using Isabelle Refinement Framework

Voting Rule	#Alternatives	#Voters	Original	Refined
Plurality	9	50 000	32.0 s	0.22 s
	20	50 000	396.0 s	0.22 s
	50	50 000	✗	0.20 s
Borda	20	50 000	395.0 s	1.30 s
	20	500	4.5 s	0.12 s
Pairwise Majority	20	500	145.0 s	0.25 s
	50	500	87.0 s	5.70 s

A Construction Pattern: Distance Rationalization (DR)

- Construction of voting rules out of consensus K and election distance d (Lerer and Nitzan, 1985)
- DR rule $\mathcal{R}(K,d)(E) := \underset{r \in \sigma(E)}{\arg\min}\ d(E, K_r) = \underset{r \in \sigma(E)}{\arg\min}\ \left(\inf_{E' \in K_r} d(E, E')\right)$

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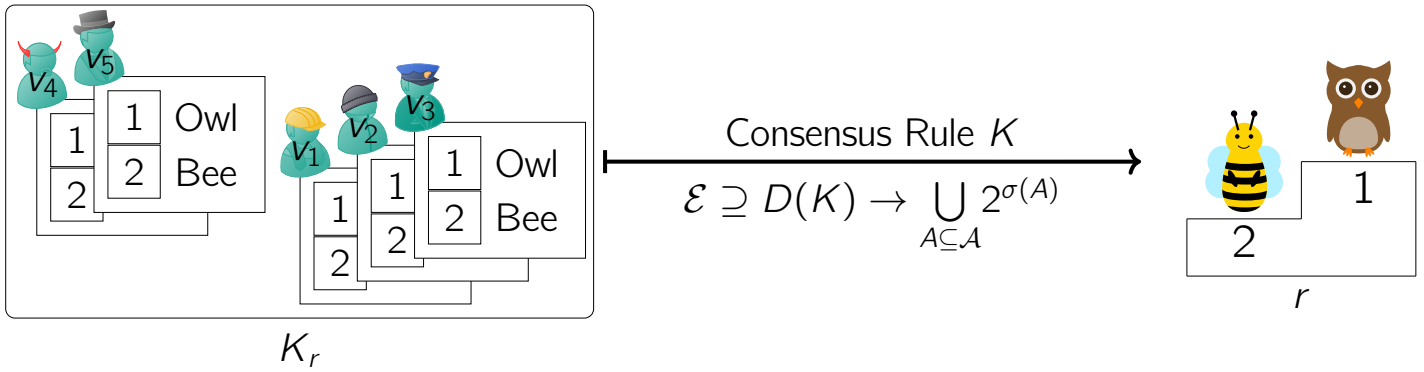

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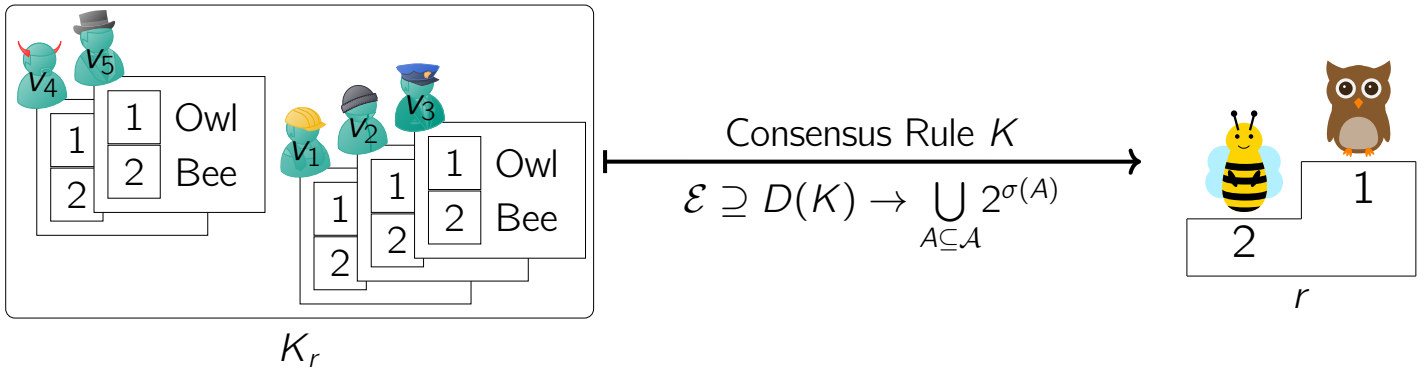

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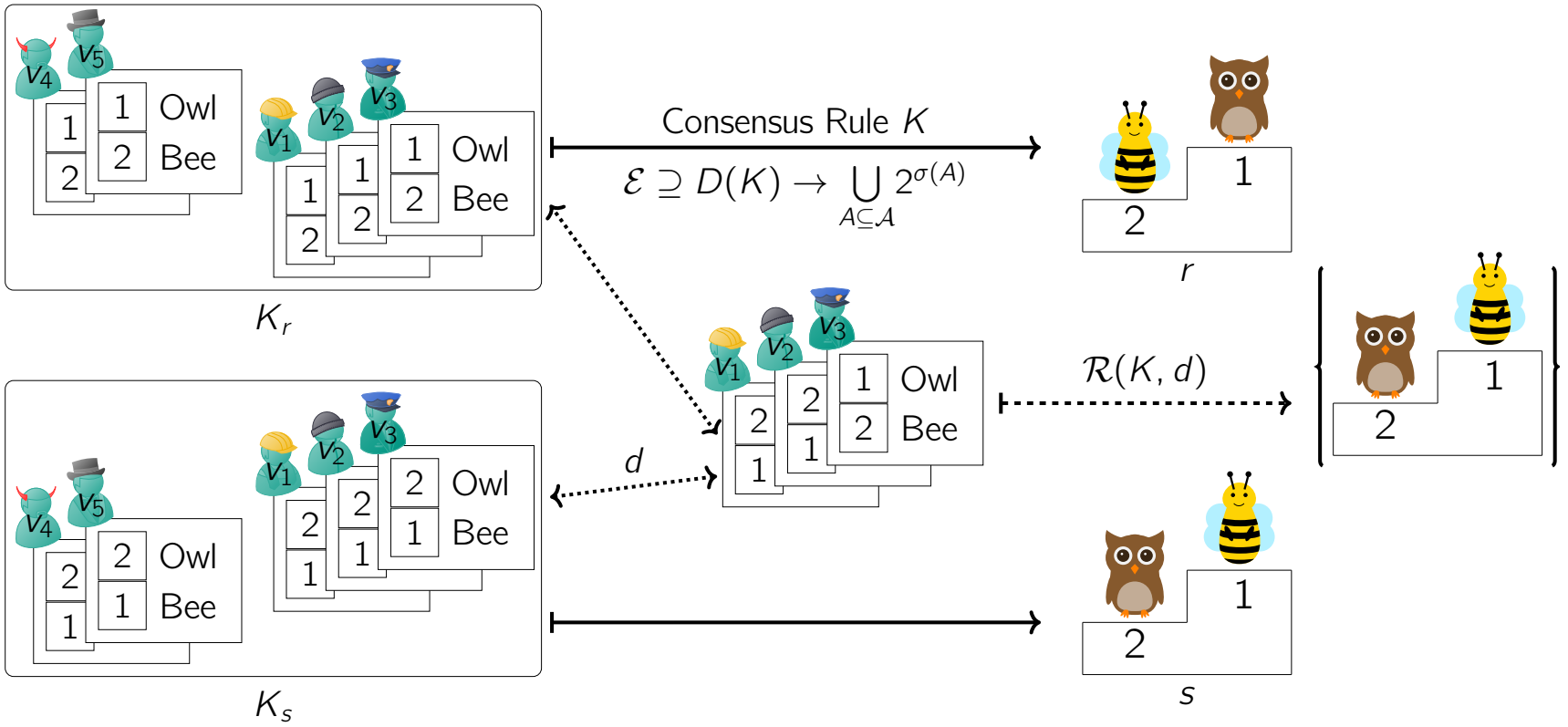

General DR-Rules for Symmetry Properties

Definition: Invariance and Equivariance

Let $f : X \to Y$, $(G, *)$ a group and ϕ, ψ be group actions of G on X respectively Y.
Then f is ϕ-**invariant** if $f(\phi(g, x)) = f(x)$ for all $x \in X$, $g \in G$.
f is (ϕ, ψ)-**equivariant** if $f(\phi(g, x)) = \psi(g, f(x))$ for all $x \in X$, $g \in G$.

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Figure: General Inference Rules (Hadjibeyli and Wilson, 2016)

Abstract Voting and Symmetries Based on Distances

fun *distance-infimum* :: *'a Distance* ⇒ *'a set* ⇒ *'a* ⇒ *ereal* **where**
distance-infimum d A a = *Inf* (*d a* ‘ *A*)

fun *closest-preimg-distance* :: (*'a* ⇒ *'b*) ⇒ *'a set* ⇒ *'a Distance* ⇒
'a ⇒ *'b* ⇒ *ereal* **where**
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fun *minimizer* :: (*'a* ⇒ *'b*) ⇒ *'a set* ⇒ *'a Distance* ⇒ *'b set* ⇒ *'a* ⇒ *'b set* **where**
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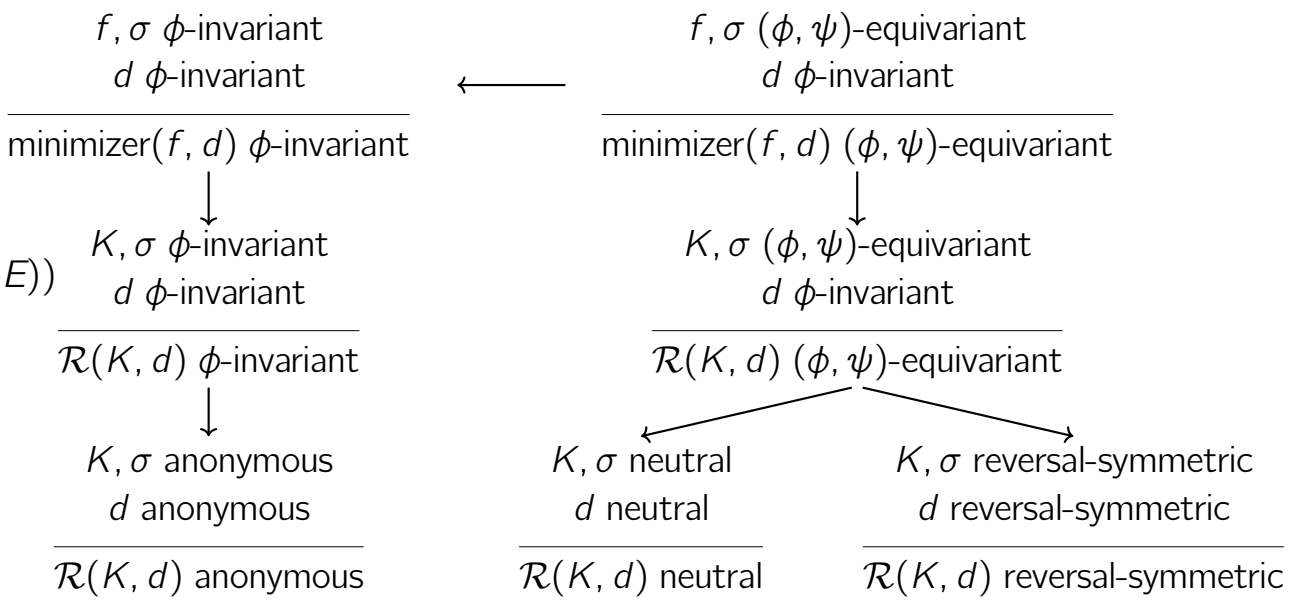

Verification of Symmetry Properties

Blueprint for Proof of Symmetry P

(1) Define P as (ϕ, ψ)-equivariance (under a group G)
   (a) Prove that G is a group
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(2) Prove that σ is (ϕ, ψ)-equivariant

(3) **Add rule for P from general inference rules.**

(4) Prove that K and d have property P (i.e., d is ϕ-invariant)
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locale *result-properties* = *result* +
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Property-Specific ("Non-Reusable") Proof Effort

Step (1): homomorphism axiom $(\phi(g, \cdot) \circ \phi(h, \cdot) = \phi(g * h, \cdot))$. Steps (2), (3), (4): Depends on σ and the group action.

Conclusion

Summary

- Devised formal methods for systematic cross-layer development of trustworthy voting systems
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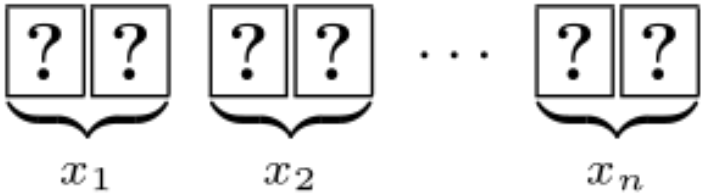
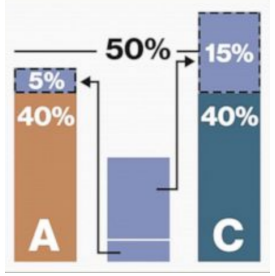
Evaluated methods on various **vote tallying algorithms:** Condorcet procedures and classical knock-out tournament

Open Research Questions

- Extension of methods for vote transfers and committee elections





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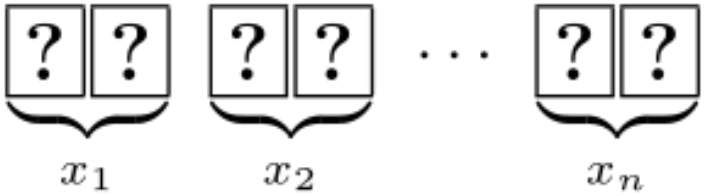

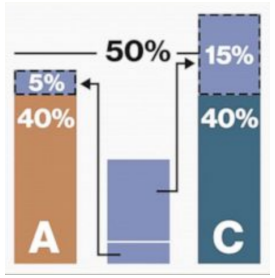

- Flexibilization of robust-by-construction framework for quantitative reasoning

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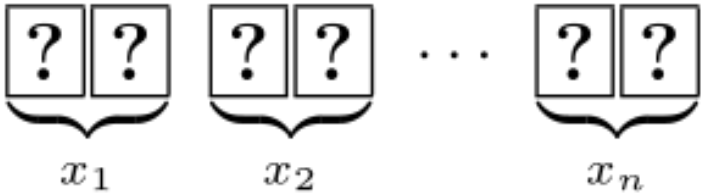

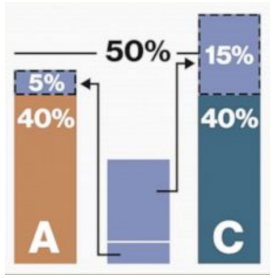

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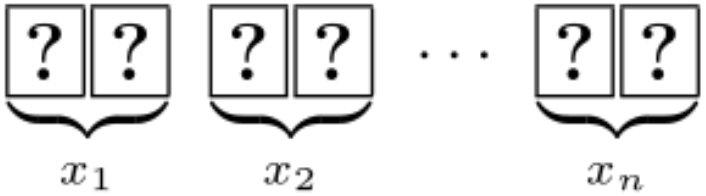
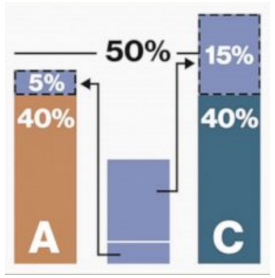
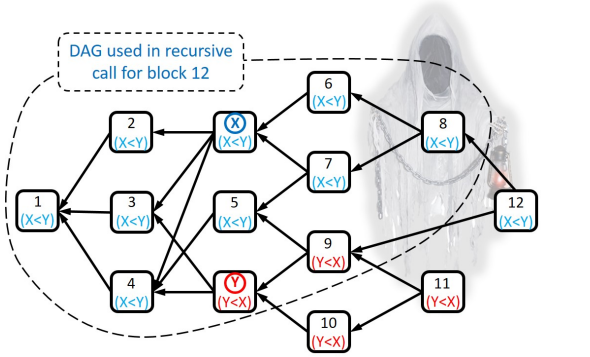




- Flexibilization of robust-by-construction framework for quantitative reasoning

- Integration of cryptographically secure computations into robust-by-construction framework

- Transfer of methods to further domains, e.g., consensus on distributed ledgers



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