Semantic Tableaux

What is the tableau method?

– Proof by contradiction, and

– case distinction until elementary contradictions are found

The example from set theory again

$$
\begin{array}{lrcl}
(1) & S \cap Q & = & \emptyset \\
(2) & P & \subseteq & Q \cup R \\
(3) & P = \emptyset & \leadsto & Q \neq \emptyset \\
(4) & Q \cup R & \subseteq & S \\
\hline
(5) & P \cap R & \neq & \emptyset
\end{array}
$$

Proof by contradiction

Assumed (1)-(4) hold but (5) does not hold

Then we have:

(3)	$c \in P$ or $d \in Q$ for some elements c, d
not (5)	$c \notin P$ or $c \notin R$
(2)	$c \notin P$ or $c \in Q$ or $c \in R$
(1)	$c \notin S$ or $c \notin Q$
(4)	$c \notin Q \cup R$ or $c \in S$

Comment:

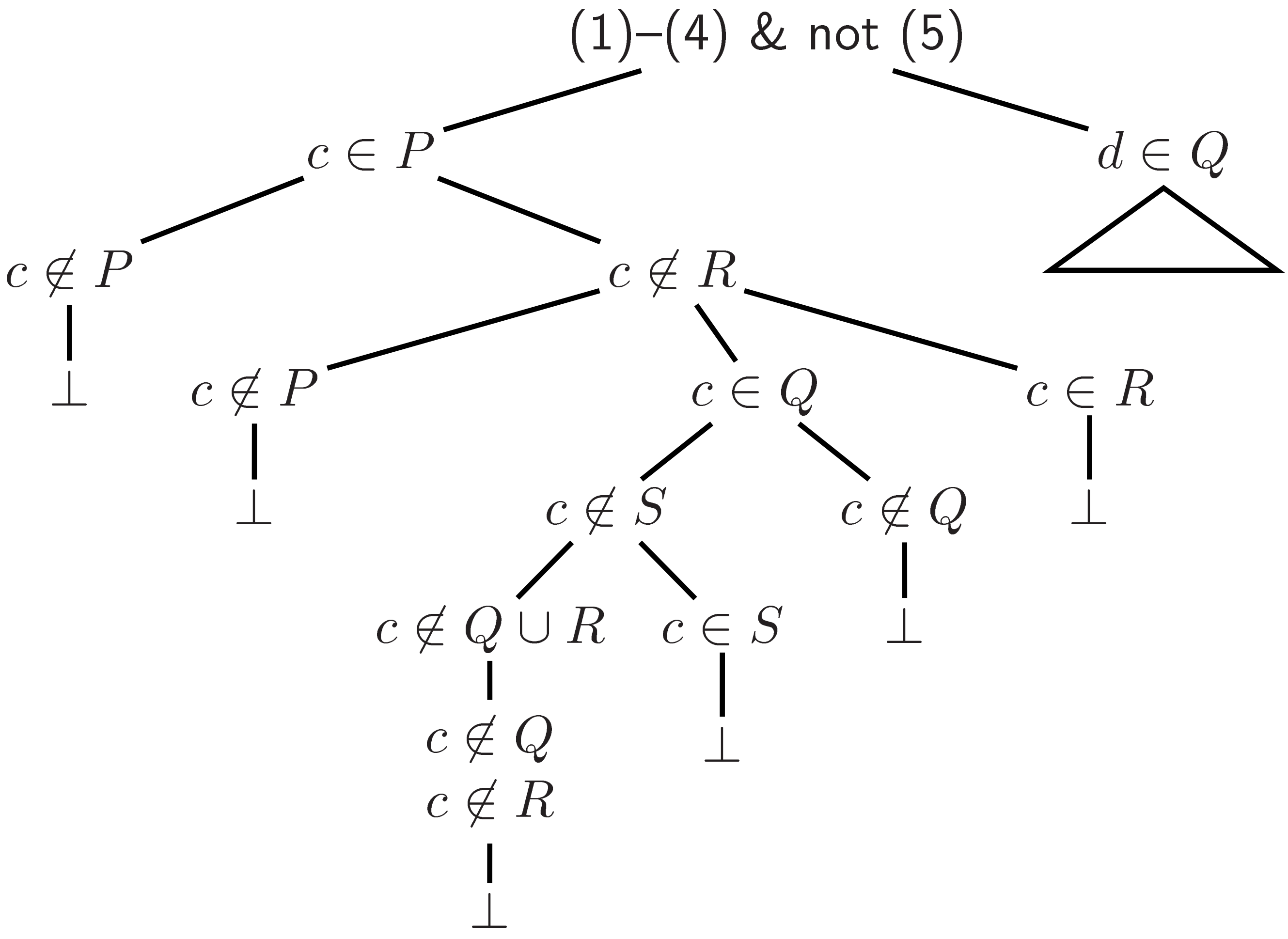
We are using instances (for c) of the more general assumptions

Example from Set Theory (cont.)

(3)	$c \in P$ or $d \in Q$ for some Elements c, d
not (5)	$c \notin P$ or $c \notin R$
(2)	$c \notin P$ or $c \in Q$ or $c \in R$
(1)	$c \notin S$ or $c \notin Q$
(4)	$c \notin Q \cup R$ or $c \in S$

Case distinction

$\perp$ symbolises contradiction



Comment:

Semantic Tableaux

Characteristic properties

– Proof by contradiction

– Proof by case distinction

– Top-down analysis of the problem

The problem of finding the right instances:

Instances of the assumptions are used in the proof.

Problem:
It is very difficult to “guess” which instances are going to be useful in the proof.

Solution:
Free variable tableaux

Use free variables (dummies, place holders) that are instantiated “on demand”.

Comment:

Top-down: case distinction according to the syntactical form of the assumptions.

In the example we used instances for c.

On demand means: by the branch closure rule ("we would have a contradiction if . . . ")

Rules of Free Variable Tableaux

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$$	conjunctive	$p \wedge q$ p q
$$\frac{\beta}{\beta_1 \mid \beta_2}$$	disjunctive	$p \vee q$ $p \quad q$
$$\frac{\gamma}{\gamma_1(y)}$$ $y \in$ Var new	universal	$\forall x(p(x))$ $p(y)$
$$\frac{\delta}{\delta_1(f_\delta(x_1, \ldots, x_n))}$$ $\{x_1, \ldots, x_n\} =$ Free(δ)	existential	$\exists x(p(x,y))$ $p(f(y))$
$$\frac{\begin{array}{c}\Psi\\ \neg\psi\end{array}}{\bot}$$ Ψ, ψ unifiable literals	contradiction	$p(x)$ $p(a)$ $\bot$
Mgu of Ψ, ψ is applied to the <u>whole</u> tableau!		$\{x \leftarrow a\}$ applied

Comment:

All rules are deterministic (the calculus is not)

The γ-rule has to be applied repeatedly, the others not. It is the reason why proof construction does not terminate.

The δ-rule uses an improved Skolemisation where each δ-formula is assigned its own unique Skolem symbol, which makes the rule deterministic; f_δ is the symbol assigned to δ.

The closure rule uses only literals. One could close branches with complementary complex formulas as well; but that is usually not useful in AD.

Tableau Proofs

$\Psi = \{\phi_1, \ldots, \phi_m\}$ a set of closed formulas

Tableaux construction rules

Initialisation

A linear tree whose m nodes are marked with $\phi_1, \ldots, \phi_m$ is a tableau for Ψ

Expansion

- T a tableau for Ψ
- B a branch of T
- B contains a premiss for one of the α-, ... , δ-rules or the closure rule

Then T' is a tableau for Ψ if obtained by applying the appropriate rule and expanding B with the formulas in the conclusion.

Closed tableau

A branch is a maximal path in the tableau tree.

A tableau is closed if all its branches are closed, i.e., contain $\perp$

Tableau proof

A closed tableau for Ψ is a tableau proof for Ψ
(for the unsatisfiability of Ψ)

Comment:

Expansion contains branch closure as a special case

Properties of Free Variable Tableaux

Ψ a set of closed formulas

Soundness

If there is a closed tableau for Ψ,
then Ψ is unsatisfiable.

Proof sketch:
All rules preserve satisfiability of tableaux.
Thus, if Ψ is satisfiable, then all tableaux for Ψ are satisfiable.
Closed tableaux are not satisfiable.
Therefore, there are no closed tableau for satisfiable Ψ.

Completeness

If Ψ is unsatisfiable,
then there is a closed tableau for Ψ.

Proof sketch:
Construct a tableau for Ψ using fairness.
Make sure that all possible rule applications are used.
If that tableau is not closed, it has at least one open (infinite) branch.
The literals on that branch define a model for Ψ.

Comment:

It is important that the formulas in Ψ are closed. Free variables now have a special meaning as they are used as place holders for terms.

<u>Remark:</u>
A tableau T is satisfiable iff there is an interpretation $\mathcal{D}, I$ such that all ground instances T^* of T have a branch B^* whose formulae are all true in $\mathcal{D}, I$.

Properties of Free Variable Tableaux (cont.)

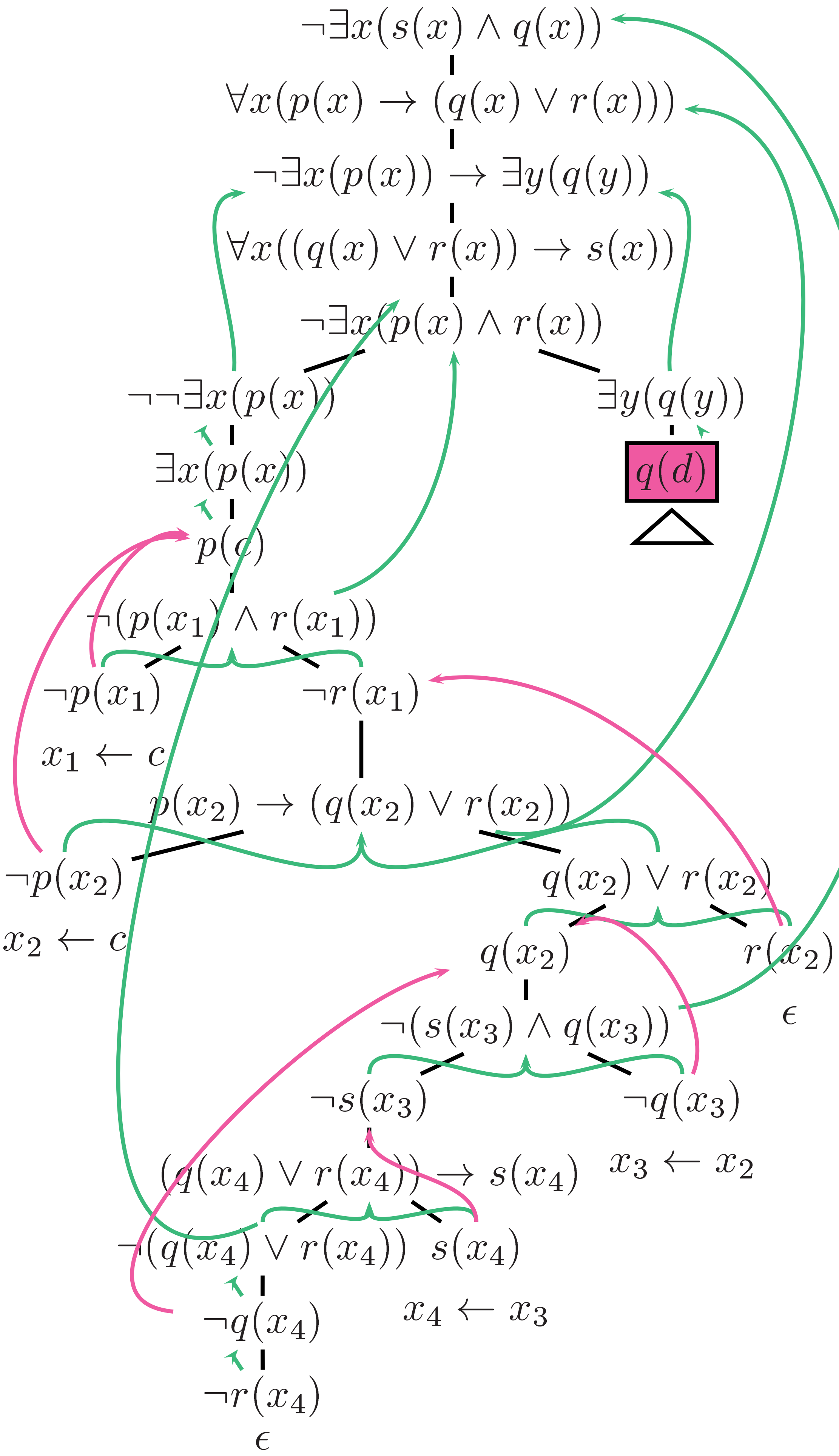
Proof confluent

If Ψ is unsatisfiable,
then every tableau for Ψ can be extended to a closed tableau.

Proof sketch:
If Ψ is unsatisfiable, then there is a closed tableau T for Ψ
Construct T at the end of each open branch of the not yet closed tableau.

Comment:

Proof confluent means that there are no dead ends in the proof search.



$\neg\exists x(s(x) \wedge q(x))$
$\forall x(p(x) \to (q(x) \vee r(x)))$
$\neg\exists x(p(x)) \to \exists y(q(y))$
$\forall x((q(x) \vee r(x)) \to s(x))$
$\neg\exists x(p(x) \wedge r(x))$
$\neg\neg\exists x(p(x))$
$\exists y(q(y))$
$\exists x(p(x))$
$q(d)$
$p(c)$
$\neg(p(x_1) \wedge r(x_1))$
$\neg p(x_1)$
$\neg r(x_1)$
$x_1 \leftarrow c$
$p(x_2) \to (q(x_2) \vee r(x_2))$
$\neg p(x_2)$
$q(x_2) \vee r(x_2)$
$x_2 \leftarrow c$
$q(x_2)$
$r(x_2)$
ϵ
$\neg(s(x_3) \wedge q(x_3))$
$\neg s(x_3)$
$\neg q(x_3)$
$(q(x_4) \vee r(x_4)) \to s(x_4)$
$x_3 \leftarrow x_2$
$\neg(q(x_4) \vee r(x_4))$
$s(x_4)$
$x_4 \leftarrow x_3$
$\neg q(x_4)$
$\neg r(x_4)$
ϵ

Comment:

The example from set theory again.