Formal Methods in Software Engineering

Modal Logic

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Modal Logic

In classical logic, it is only important whether a formula is true

In modal logic, it is also important in which

- **way**
- **mode**
- **state**

a formula is true

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In modal logic, it is also important in which

- **way**
- **mode**
- **state**

a formula is true

A formula (a proposition) is

- **necessarily / possibly true**
- **true today / tomorrow**
- **believed / known**
- **true before / after an action / the execution of a program**

Propositional Modal Logic: Formulas

- **The propositional variables $p \in$ Var are modal formulas**
- **If A, B are modal formulas, then**

$$\neg A \qquad (A \land B) \qquad (A \lor B) \qquad (A \rightarrow B) \qquad (A \leftrightarrow B)$$

$\Box A$ **(read "box A", "necessarily A")**

$\Diamond A$ **(read "diamond A", "possibly A")**

are modal formulas

Informal Interpretations of $\Box$

$\Box F$ means

- F **is necessarily true**
- F **is always true (in future states/words)**
- **an agent** a **believes** F
- **an agent** a **knows** F
- F **is true after all possible executions of a program** p

Informal Interpretations of $\Box$

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Notation

If necessary write

$$\Box_a F \qquad \Box_p F \qquad [a]F \qquad [p]F$$

instead of $\Box F$

Informal Interpretations of $\Diamond$

$\Box F$	$\Diamond F$ (the same as $\neg\Box\neg F$)
F is necessarily true	F is possibly true
F is always true	F at least once true
agent a believes F	F is consistent with a's beliefs
agent a knows F	a does not know $\neg F$
F is true after all possible executions of program p	F is true after at least one possible execution of program p

Kripke Structures

Given: a propositional signature Var

Definition

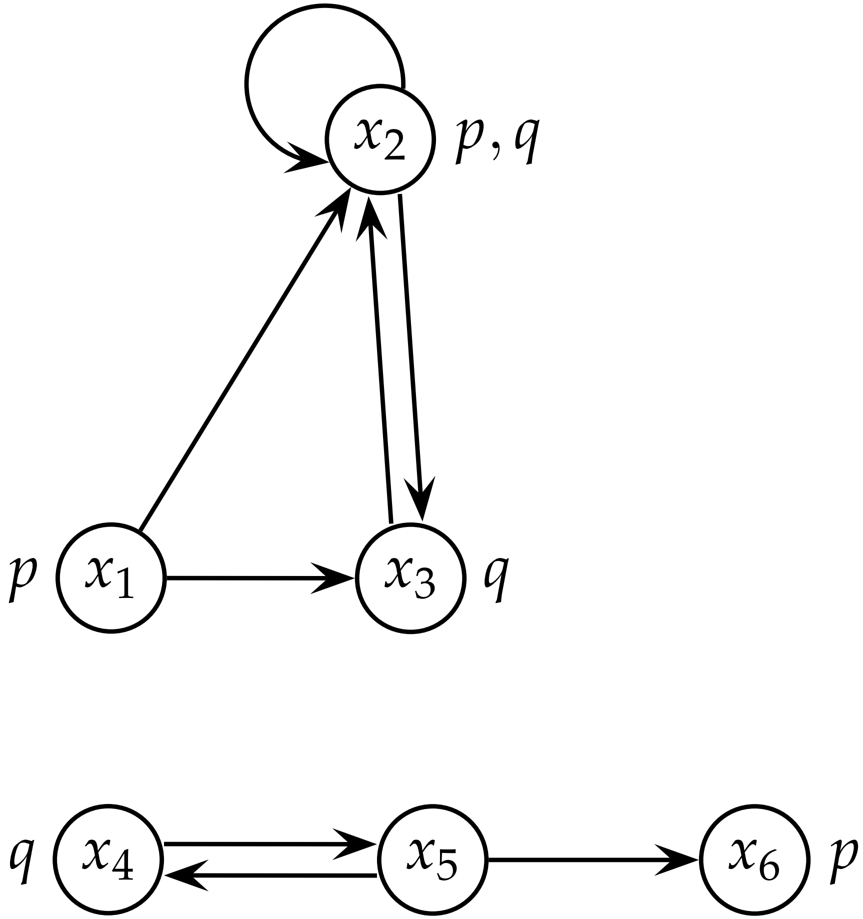
A Kripke structure

$$\mathcal{K} = (S, R, I)$$

consists of

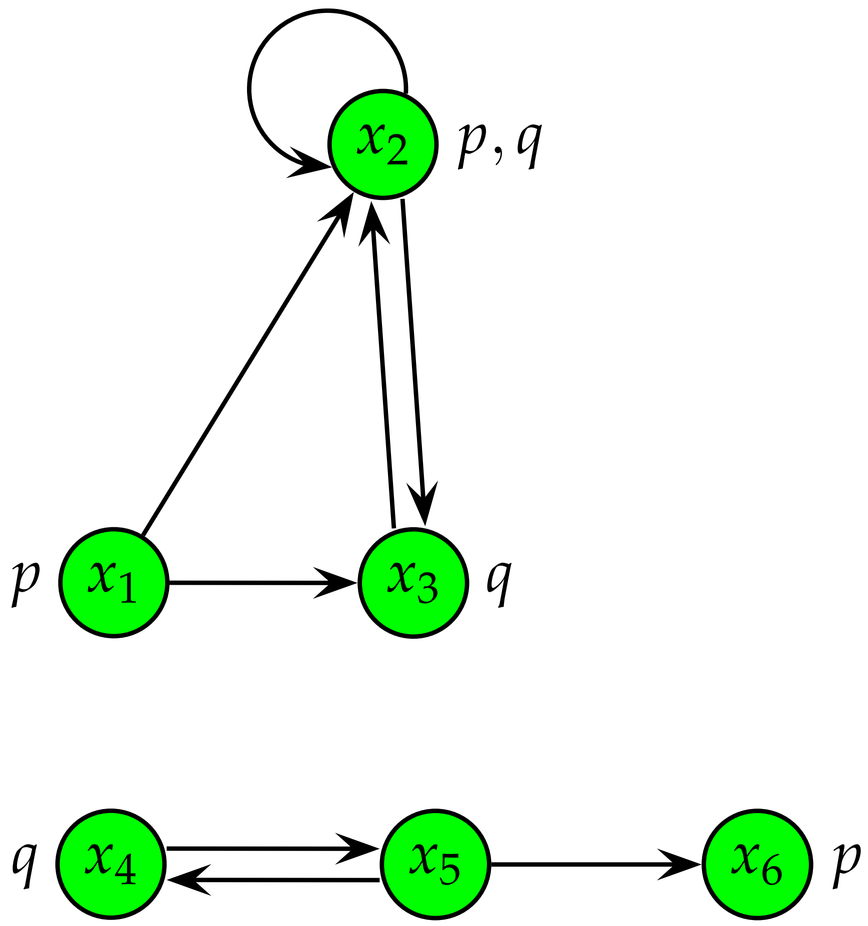
- **a non-empty set** S **(of worlds / states)**
- **an *accessibility relation*** $R \subseteq S \times S$
- **an *interpretation*** I : **Var** $\times S \rightarrow \{\underline{\textbf{true}}, \underline{\textbf{false}}\}$

Kripke Structures: Example



Kripke Structures: Example

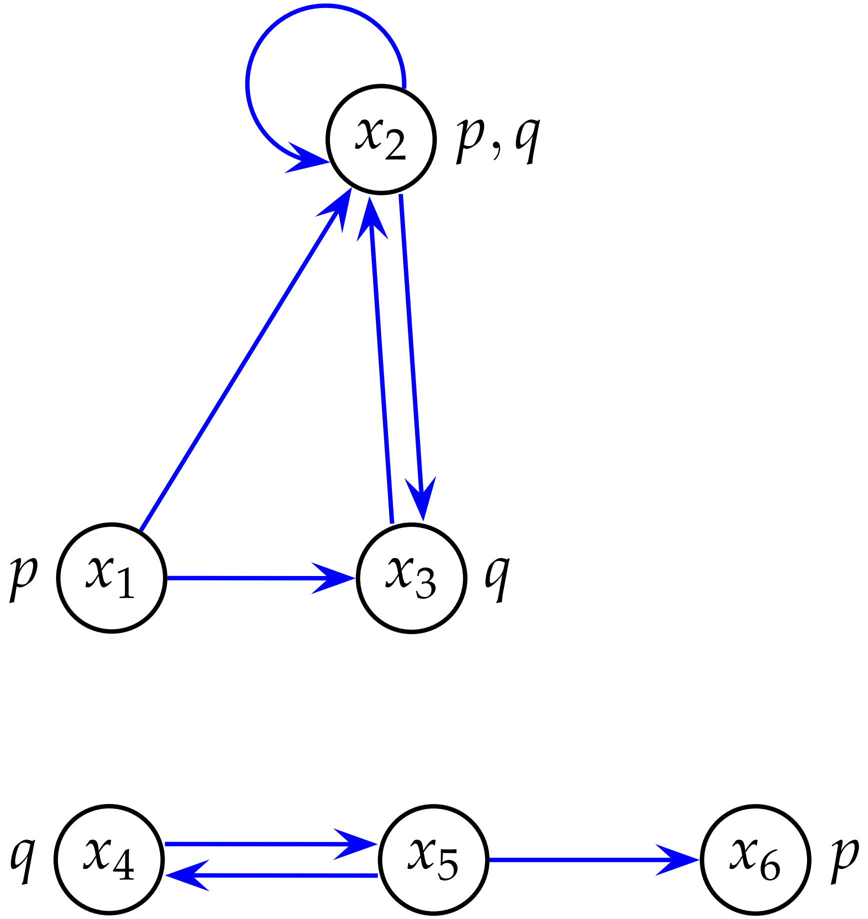
set of states



Kripke Structures: Example

accessibility relation

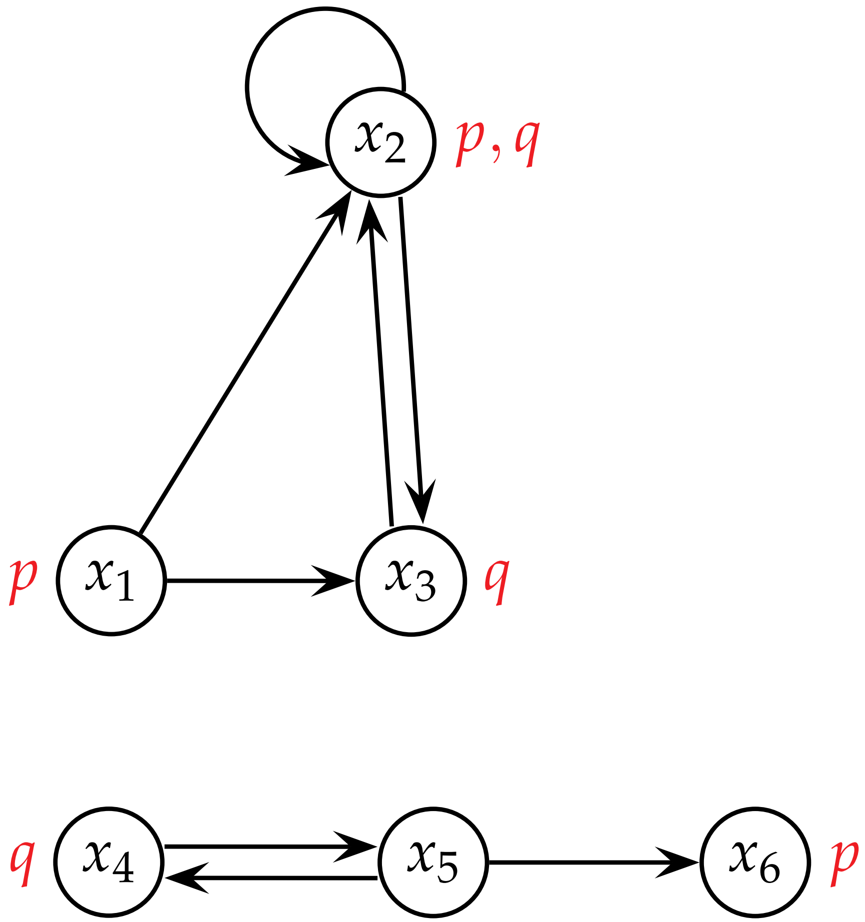
set of states



Kripke Structures: Example

accessibility relation

set of states



Interpretation I

Modal Logic: Semantics

Given: Kripke structure $\mathcal{K} = (S, R, I)$

Valuation

$val_{\mathcal{K}}(p)(s) \;=\; I(p)(s)$ **for** $p \in$ **Var**

$val_{\mathcal{K}}$ **defined for propositional operators in the same way as val**$_I$

$$val_{\mathcal{K}}(\Box A)(s) \;=\; \begin{cases} \underline{\textbf{true}} & \textbf{if } val_{\mathcal{K}}(A)(s') = \underline{\textbf{true}} \textbf{ for} \\ & \textbf{all } s' \in S \textbf{ with } sRs' \\ \underline{\textbf{false}} & \textbf{otherwise} \end{cases}$$

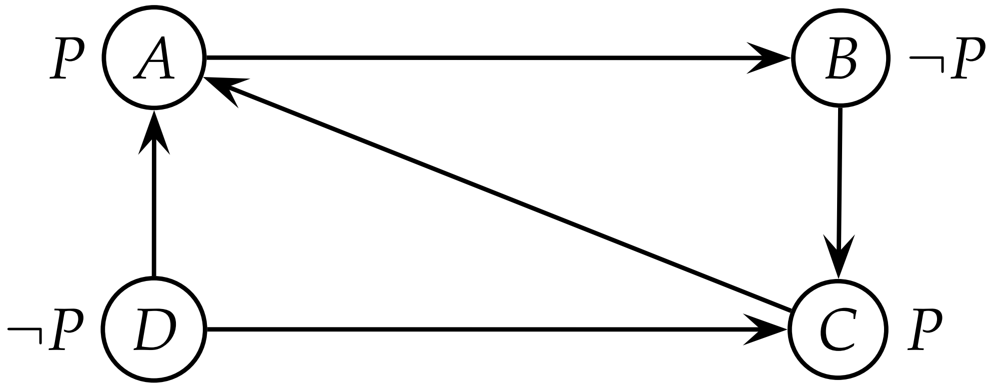
$$val_{\mathcal{K}}(\Diamond A)(s) \;=\; \begin{cases} \underline{\textbf{true}} & \textbf{if } val_{\mathcal{K}}(A)(s') = \underline{\textbf{true}} \textbf{ for} \\ & \textbf{at least one } s' \in S \textbf{ with } sRs' \\ \underline{\textbf{false}} & \textbf{otherwise} \end{cases}$$

Saul Aaron Kripke

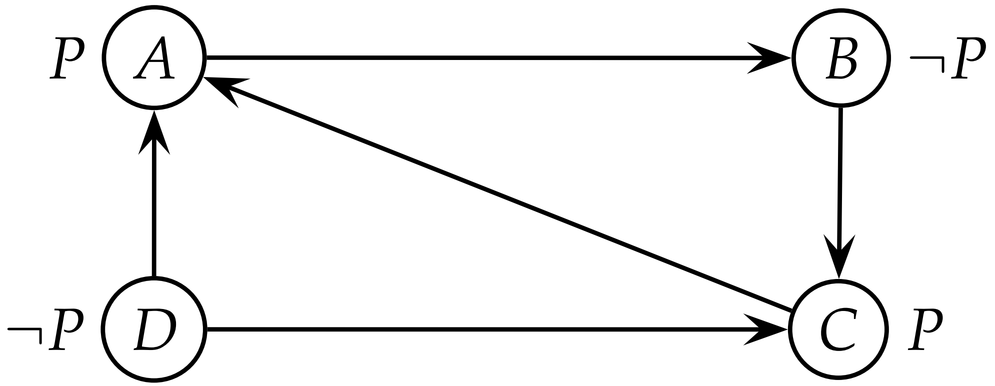
Born 1940 in Omaha (US)

First publication:	*A Completeness Theorem in Modal Logic* **The Journal of Symbolic Logic, 1959**
Studied at:	**Harvard, Princeton, Oxford and Rockefeller University**
Positions:	**Harvard, Rockefeller, Columbia, Cornell, Berkeley, UCLA, Oxford**
since 1977	**Professor at Princeton University**
since 1998	**Emeritus at Princeton University**

Modal Logic: Example for Evaluation



Modal Logic: Example for Evaluation



$(\mathcal{K}, A) \models P$	$(\mathcal{K}, B) \models \neg P$	$(\mathcal{K}, C) \models P$	$(\mathcal{K}, D) \models \neg P$
$(\mathcal{K}, A) \models \Box \neg P$	$(\mathcal{K}, B) \models \Box P$	$(\mathcal{K}, C) \models \Box P$	$(\mathcal{K}, D) \models \Box P$
$(\mathcal{K}, A) \models \Box\Box P$	$(\mathcal{K}, B) \models \Box\Box P$	$(\mathcal{K}, C) \models \Box\Box \neg P$	—

Modal Logic: Valid Formulas

Valid

- $\Box(P \to Q) \to (\Box P \to \Box Q)$
- $(\Box P \land \Box(P \to Q)) \to \Box Q$
- $(\Box P \lor \Box Q) \to \Box(P \lor Q)$
- $(\Box P \land \Box Q) \leftrightarrow \Box(P \land Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \lor Q) \leftrightarrow (\Diamond P \lor \Diamond Q)$
- $\Diamond(P \land Q) \to (\Diamond P \land \Diamond Q)$

Modal Logic: Valid Formulas

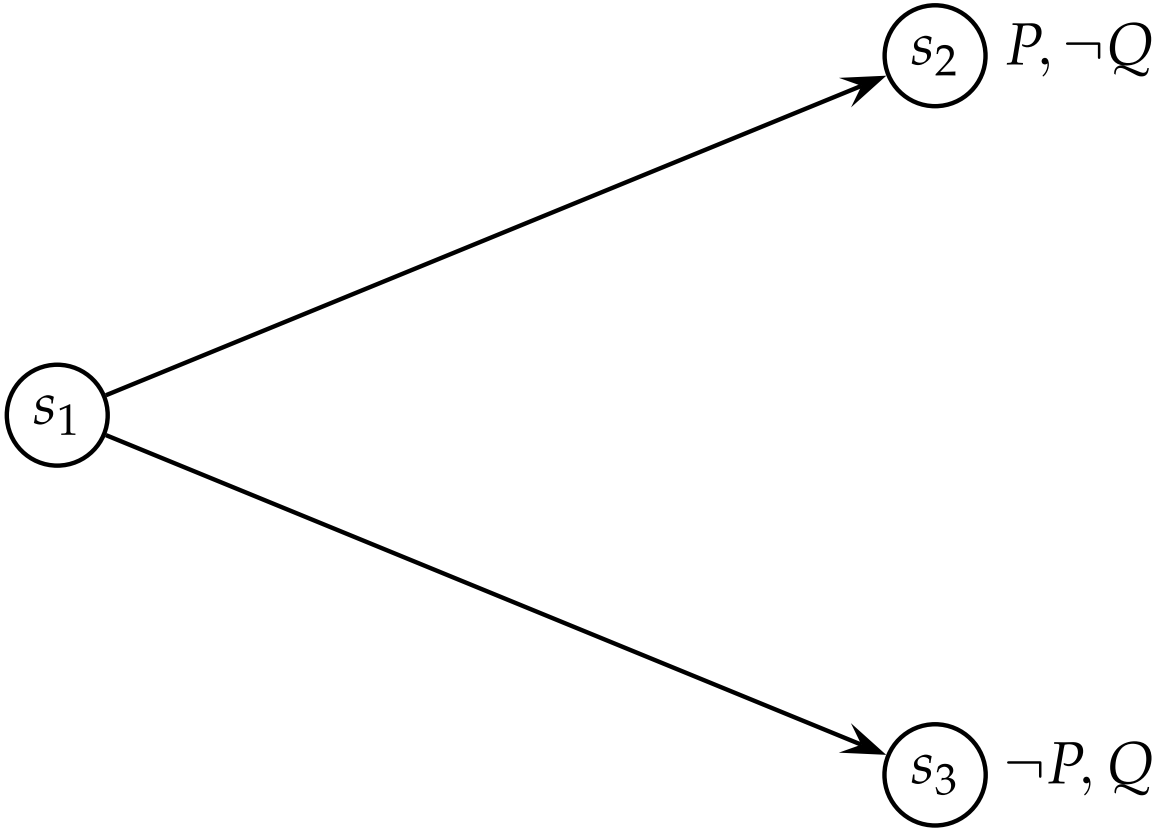
Valid

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \land \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \lor \Box Q) \rightarrow \Box(P \lor Q)$
- $(\Box P \land \Box Q) \leftrightarrow \Box(P \land Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \lor Q) \leftrightarrow (\Diamond P \lor \Diamond Q)$
- $\Diamond(P \land Q) \rightarrow (\Diamond P \land \Diamond Q)$

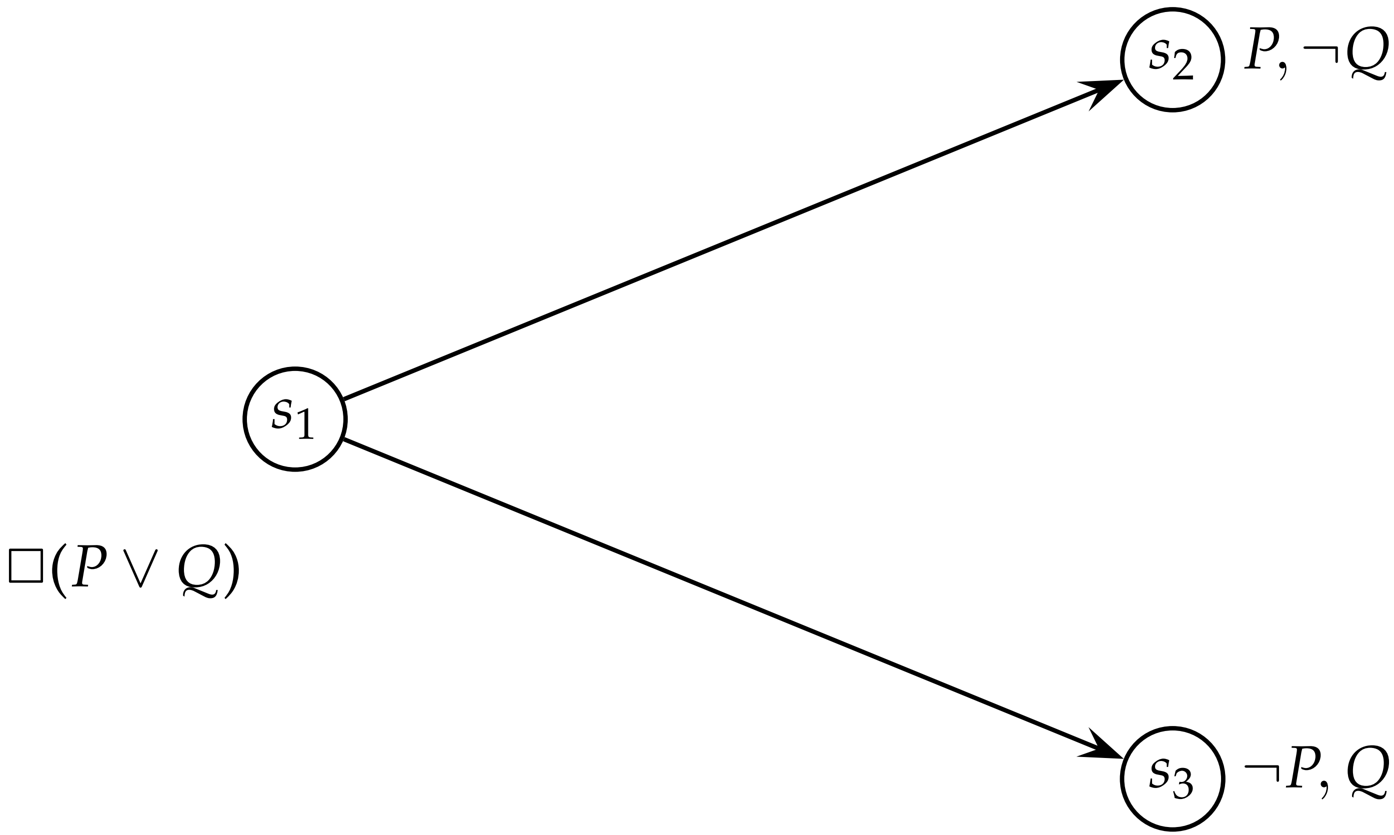
Not valid:

- $\Box(P \lor Q) \rightarrow (\Box P \lor \Box Q)$
- $(\Diamond P \land \Diamond Q) \rightarrow \Diamond(P \land Q)$

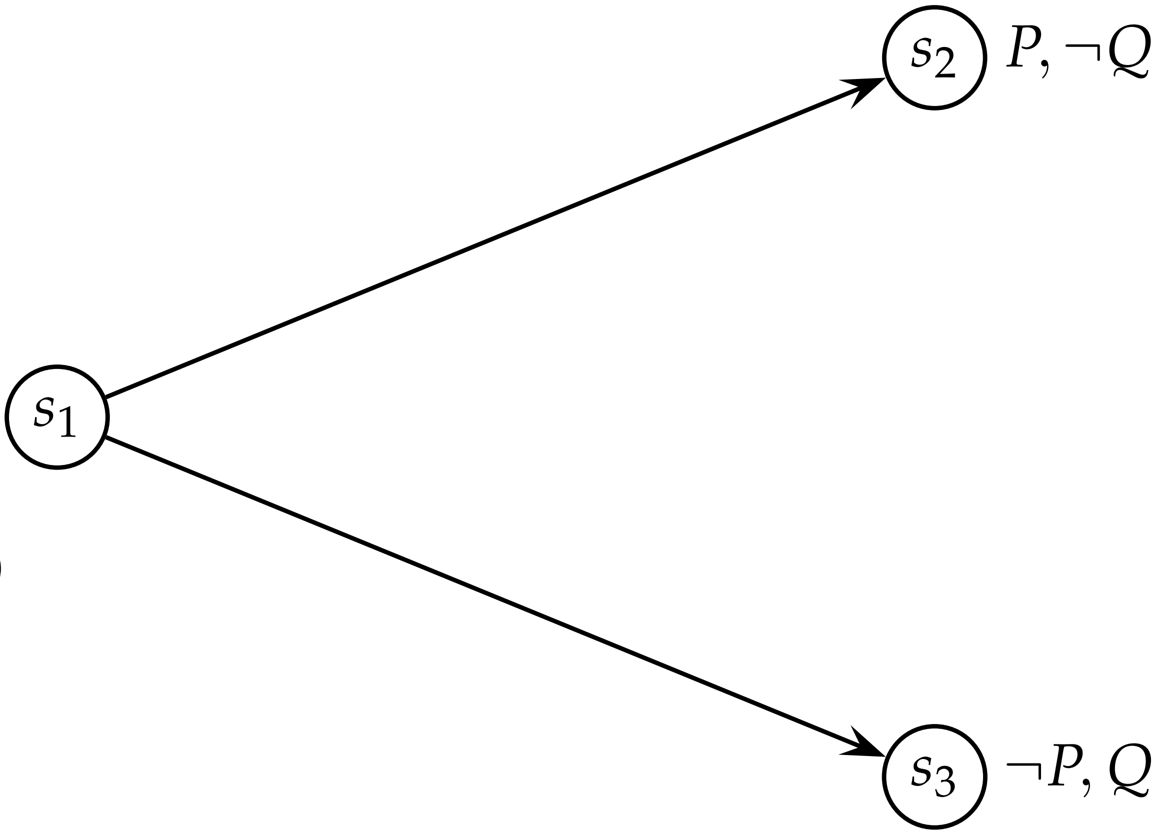
Not Valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$



Not Valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

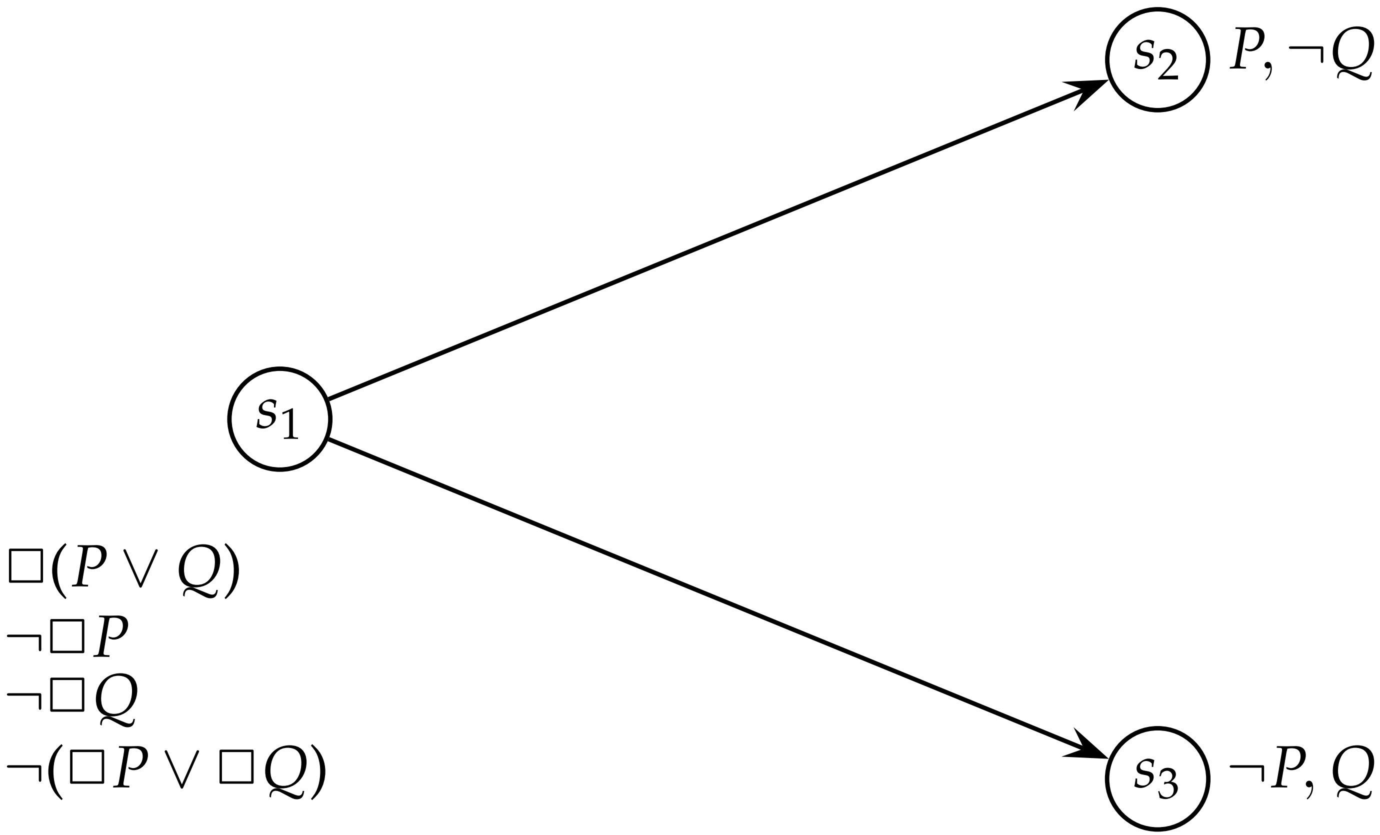


Not Valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

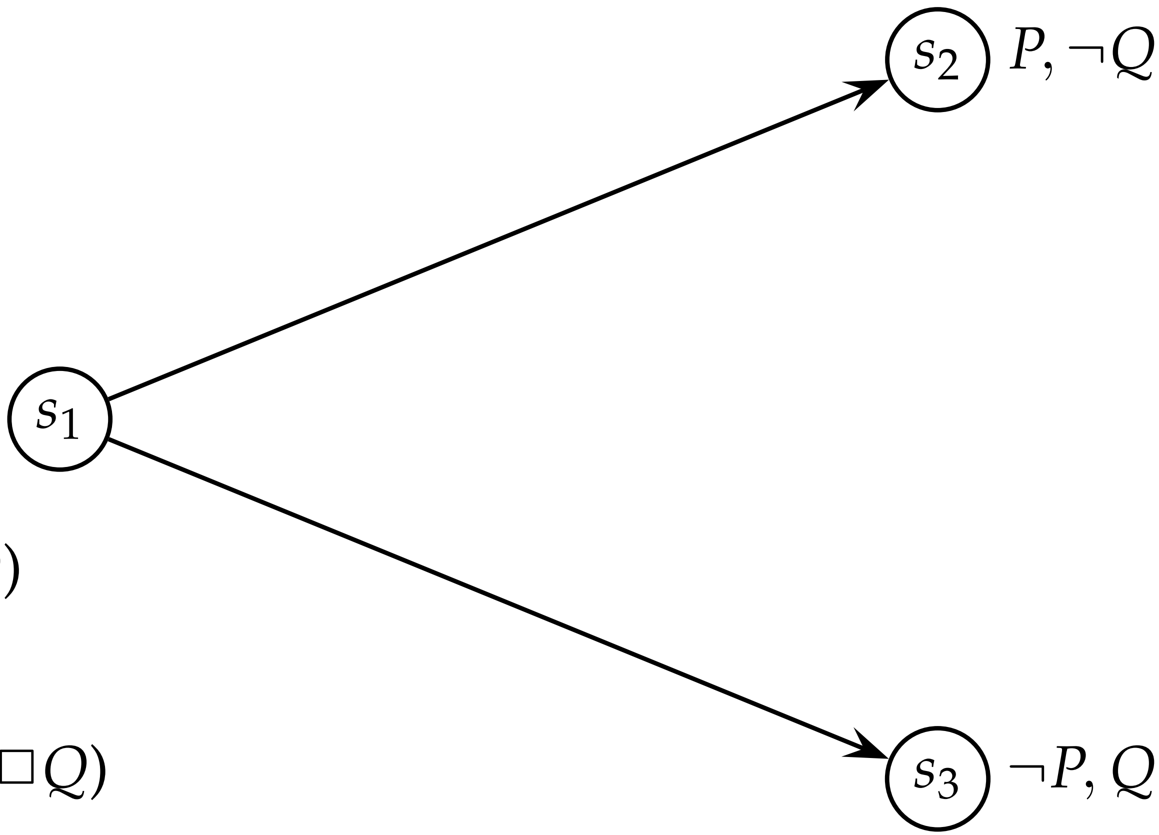


$\Box(P \vee Q)$
$\neg\Box P$
$\neg\Box Q$

Not Valid: $\Box(P \lor Q) \to (\Box P \lor \Box Q)$



Not Valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$



$\Box(P \vee Q)$
$\neg\Box P$
$\neg\Box Q$
$\neg(\Box P \vee \Box Q)$

$\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$ **not true in state** s_1

Formulas Characterising Properties of R

Formula	**Property of R**
$\Box p \rightarrow p$	**reflexive**
$p \rightarrow \Diamond p$	**reflexive**
$\Box\Box p \rightarrow \Box p$	**reflexive**
$\Box\Diamond p \rightarrow \Diamond p$	**reflexive**
$\Box p \rightarrow \Diamond\Box p$	**reflexive**
$\Diamond\Diamond p \rightarrow \Diamond p$	**reflexive**

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$\Box p \rightarrow \Diamond\Box p$	**reflexive**
$\Diamond\Diamond p \rightarrow \Diamond p$	**reflexive**

Formula	Property of R
$\Box p \rightarrow \Box\Box p$	**transitive**
$p \rightarrow \Box\Diamond p$	**symmetrical**
$\Box\Box p \leftrightarrow \Box p$	**reflexive, transitive**
$\Diamond\Diamond p \leftrightarrow \Diamond p$	**reflexive, transitive**
$\Diamond\Box p \leftrightarrow \Box p$	**equivalence relation**
$\Box\Diamond p \leftrightarrow \Diamond p$	**equivalence relation**

Modal Logic: Valid Formulas

$\Box F$	$\Box F \to F$	$\Box F \to \Box\Box F$	$\Box F \to \Diamond F$	$(\Box(F \to G) \land \Box F) \to \Box G$	$\Diamond true$
F **is necessarily true**					
agent a **knows** F					
agent a **believes** F					
F **holds after executing program** p					

Modal Logic: Valid Formulas

$\Box F$	$\Box F \to F$	$\Box F \to \Box\Box F$	$\Box F \to \Diamond F$	$(\Box(F \to G) \wedge \Box F) \to \Box G$	$\Diamond true$
F **is necessarily true**	yes	yes	yes	yes	yes
agent a **knows** F	yes	yes	yes	yes	yes
agent a **believes** F	no	yes	yes	yes	yes
F **holds after executing program** p	no	no	no	yes	no