Formal Verification of Software

Propositional and Predicate Logic

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Propositional Logic: Syntax

Special symbols

$$(\quad)\quad \neg\quad \wedge\quad \vee\quad \rightarrow\quad \leftrightarrow$$

Propositional Logic: Syntax

Special symbols

$($ $)$ $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$

Signature

Propositional variables $\Sigma = \{p_0, p_1, \ldots\}$

Propositional Logic: Syntax

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$($ $)$ $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$

Signature

Propositional variables $\Sigma = \{p_0, p_1, \ldots\}$

Formulas

- **The propositional variables** $p \in \Sigma$ **are formulas**
- **If** A, B **are formulas, then**

$$\neg A \qquad (A \wedge B) \qquad (A \vee B) \qquad (A \rightarrow B) \qquad (A \leftrightarrow B)$$

are formulas

Propositional Logic: Unified Notation

Introduced by Smullyan, 1968

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Conjunctive formulas **Type** α

$$\neg\neg A \qquad (A \wedge B) \qquad \neg(A \vee B) \qquad \neg(A \rightarrow B)$$

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Conjunctive formulas **Type α**

$$\neg\neg A \qquad (A \wedge B) \qquad \neg(A \vee B) \qquad \neg(A \rightarrow B)$$

Disjunctive formulas **Type β**

$$\neg(A \wedge B) \qquad (A \vee B) \qquad (A \rightarrow B)$$

Propositional Logic: Unified Notation

Non-literal formulas and their corresponding "logical" sub-formulas

α	α_1	α_2
$A \wedge B$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \rightarrow B)$	A	$\neg B$
$\neg\neg A$	A	A

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$\neg(A \rightarrow B)$	A	$\neg B$
$\neg\neg A$	A	A

β	β_1	β_2
$\neg(A \wedge B)$	$\neg A$	$\neg B$
$A \vee B$	A	B
$A \rightarrow B$	$\neg A$	B

Propositional Logic: Semantics

Interpretation

Function $I : \Sigma \rightarrow \{\underline{\textbf{true}}, \underline{\textbf{false}}\}$

Propositional Logic: Semantics

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Function $I : \Sigma \rightarrow \{\underline{\textbf{true}}, \underline{\textbf{false}}\}$

Valuation

Extension of interpretation to formulas as follows:

$$\textbf{val}_I(p) \quad = \quad I(p)$$

$$\textbf{val}_I(\neg p) \quad = \quad \begin{cases} \underline{\textbf{true}} & \textbf{if } I(p) = \underline{\textbf{false}} \\ \underline{\textbf{false}} & \textbf{if } I(p) = \underline{\textbf{true}} \end{cases}$$

Propositional Logic: Semantics

$$\textbf{val}_I(\alpha) = \begin{cases} \underline{\textbf{true}} & \textbf{if val}_I(\alpha_1) = \underline{\textbf{true}} \\ & \textbf{and val}_I(\alpha_2) = \underline{\textbf{true}} \\ \underline{\textbf{false}} & \textbf{if val}_I(\alpha_1) = \underline{\textbf{false}} \\ & \textbf{or val}_I(\alpha_2) = \underline{\textbf{false}} \end{cases}$$

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$$\mathbf{val}_I(\beta) = \begin{cases} \underline{\mathbf{true}} & \text{if } \mathbf{val}_I(\beta_1) = \underline{\mathbf{true}} \\ & \text{or } \mathbf{val}_I(\beta_2) = \underline{\mathbf{true}} \\ \underline{\mathbf{false}} & \text{if } \mathbf{val}_I(\beta_1) = \underline{\mathbf{false}} \\ & \text{and } \mathbf{val}_I(\beta_2) = \underline{\mathbf{false}} \end{cases}$$

Propositional Logic: Semantics

$$\textbf{val}_I(A \leftrightarrow B) \quad = \quad \begin{cases} \underline{\textbf{true}} & \textbf{if } \textbf{val}_I(A) = \textbf{val}_I(B) \\ \underline{\textbf{false}} & \textbf{if } \textbf{val}_I(A) \neq \textbf{val}_I(B) \end{cases}$$

Predicate Logic: Syntax

Additional special symbols

“,” “$\forall$” “$\exists$”

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Object variables

Var $= \{x_0, x_1, \ldots\}$

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Object variables

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Signature

Triple $\Sigma = \langle F_\Sigma, P_\Sigma, \alpha_\Sigma \rangle$ **consisting of**

- **set** F_Σ **of functions symbols**
- **set** P_Σ **of predicate symbols**
- **function** $\alpha_\Sigma : F_\Sigma \cup P_\Sigma \to \mathbb{N}$
  assigning aritys to function and predicate symbols

Predicate Logic: Syntax

Terms

- **variables $x \in$ Var are terms**
- **if $f \in F_\Sigma$, $\alpha_\Sigma(f) = n$, and $t_1, \ldots, t_n$ terms, then $f(t_1, \ldots, t_n)$ is a term**

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Atoms

If $p \in P_\Sigma$, $\alpha_\Sigma(p) = n$, and $t_1, \ldots, t_n$ terms, then $p(t_1, \ldots, t_n)$ is an atom

Predicate Logic: Syntax

Formulas

- **Atoms are formulas**
- **If A, B are formulas, $x \in$ Var, then**

$$\neg A, \quad (A \wedge B), \quad (A \vee B), \quad (A \rightarrow B), \quad (A \leftrightarrow B), \quad \forall x A, \quad \exists x A$$

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are formulas

Literals

If A is an atom, then A and $\neg A$ are literals

Example

Signature

$\Sigma_{\leq} = \langle \{0, a, b, f\}, \{in\_iv, leq\}, \alpha \rangle$

with

$\alpha(0) = \alpha(a) = \alpha(b) = 0$

$\alpha(f) = 1$

$\alpha(leq) = 2$

$\alpha(in\_iv) = 3$ **(in interval)**

Example

Signature

$\Sigma_{\leq} = \langle \{0, a, b, f\}, \{in\_iv, leq\}, \alpha \rangle$

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$\alpha(0) = \alpha(a) = \alpha(b) = 0$

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$\alpha(in\_iv) = 3$ **(in interval)**

Formula

$$\phi = \neg \underbrace{leq(y, x)}_{\textbf{Atom}} \rightarrow \exists z \underbrace{(\neg leq(z, x) \wedge \neg leq(y, z))}_{\textbf{Scope of } \exists z}$$

Predicate Logic: Unified Notation

Extension of unified notation for propositional logic

Universal formulas **Type γ**

$$\forall x A \qquad \neg\exists x A$$

Predicate Logic: Unified Notation

Extension of unified notation for propositional logic

Universal formulas **Type γ**

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Existential formulas **Type δ**

$$\neg\forall x A \qquad \exists x A$$

Predicate Logic: Unified Notation

γ- and δ-formulas and their corresponding "logical" sub-formulas

γ	$\gamma_1(x)$
$\forall x A(x)$	$A(x)$
$\neg\exists x A(x)$	$\neg A(x)$

δ	$\delta_1(x)$
$\neg\forall x A(x)$	$\neg A(x)$
$\exists x A(x)$	$A(x)$

Predicate Logic: Semantics

Interpretation

A pair $\mathcal{D} = \langle D, I \rangle$ **where**

- D **an arbitrary non-empty set, the *universe***
- I **an interpretation function**

  for $f \in F_\Sigma$**:** $I(f) : D^{\alpha(f)} \to D$

  for $p \in P_\Sigma$**:** $I(p) : D^{\alpha(p)} \to \{\underline{\textbf{true}}, \underline{\textbf{false}}\}$

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Variable assignment

A function $\lambda :$ **Var** $\to D$

Predicate Logic: Semantics

Valuation

Extension of interpretation and variable assignment to formulas

$\textbf{val}_{\mathcal{D},\lambda}(x) = \lambda(x)$ **for** $x \in$ **Var**

$\textbf{val}_{\mathcal{D},\lambda}(f(t_1, \ldots, t_n)) = I(f)(\textbf{val}_{\mathcal{D},\lambda}(t_1), \ldots, \textbf{val}_{\mathcal{D},\lambda}(t_n))$

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$\textbf{val}_{\mathcal{D},\lambda}(p(t_1,\ldots,t_n)) = I(p)(\textbf{val}_{\mathcal{D},\lambda}(t_1),\ldots,\textbf{val}_{\mathcal{D},\lambda}(t_n))$

$$\textbf{val}_{\mathcal{D},\lambda}(\forall x A) = \begin{cases} \underline{\textbf{true}} & \textbf{if val}_{\mathcal{D},\lambda^d_x}(A) = \underline{\textbf{true}} \quad \textbf{for all } d \in D \\ \underline{\textbf{false}} & \textbf{otherwise} \end{cases}$$

$$\textbf{val}_{\mathcal{D},\lambda}(\exists x A) = \begin{cases} \underline{\textbf{true}} & \textbf{if val}_{\mathcal{D},\lambda^d_x}(A) = \underline{\textbf{true}} \quad \textbf{for some } d \in D \\ \underline{\textbf{false}} & \textbf{otherwise} \end{cases}$$

$\textbf{val}_{\mathcal{D},\lambda}$ **defined for propositional operators in the same way as** $\textbf{val}_I$**.**

Predicate Logic: Semantics

Example

$$
\begin{array}{lll}
D & = & \mathbb{R} \\
I(0) & = & 0 \\
I(a) & = & -1 \\
I(b) & = & 1 \\
I(f) & = & \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases} \\
I(leq) = \underline{\textbf{true}} & \textbf{iff} & x \leq_{\mathbb{R}} y \\
I(in\_iv) = \underline{\textbf{true}} & \textbf{iff} & x \in [a, b]
\end{array}
$$

Predicate Logic: Semantics

Model

An interpretation $\mathcal{D}$ is model of a set Φ of formulas iff

$\text{val}_{\mathcal{D},\lambda}(A) =$ <u>true</u> for all λ and all $A \in \Phi$.

<u>Notation:</u> $\mathcal{D} \models \Phi$

Predicate Logic: Semantics

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Satisfiable

Φ ist satisfiable iff there are

an interpretation $\mathcal{D}$ and a variable assignment λ s.t.

$\text{val}_{\mathcal{D},\lambda}(A) = \underline{\text{true}}$ for all $A \in \Phi$

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Validity

A is valid iff

all interpretations are a model of A

Predicate Logic: Semantics

Consequence

A formula A is a consequence of Φ iff

all models of Φ are models of A as well

Notation: $\Phi \models A$

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Equivalent formulas

Two formulas are equivalent iff

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Predicate Logic: Semantics

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Notation: $\Phi \models A$

Equivalent formulas

Two formulas are equivalent iff

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Satisfiability equivalent formulas

Two formulas are satisfiability equivalent iff

they are either both satisfiable or both unsatisfiable

Substitutions

Substitution

Function $\sigma : \textbf{Var} \to \mathrm{Term}$

Written as: $\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$

where
$$\sigma(x) = \begin{cases} t_i & \textbf{if } x = x_i \textbf{ for } 1 \leq i \leq n \\ x & \textbf{otherwise} \end{cases}$$

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Extension to terms and formulas

By replacing all *free* occurrences of variables x by $\sigma(x)$

Substitutions

Example:

$$
\begin{aligned}
\phi &= \neg leq(y,x) \rightarrow \exists z(\neg leq(z,x) \wedge \neg leq(y,z)) \\
\sigma &= \{x \leftarrow a,\ y \leftarrow w,\ z \leftarrow c\} \\
\phi\sigma &= \neg leq(w,a) \rightarrow \exists z(\neg leq(z,a) \wedge \neg leq(w,z))
\end{aligned}
$$

Substitutions

Example:

$$\begin{aligned} \phi &= \neg leq(y,x) \rightarrow \exists z(\neg leq(z,x) \wedge \neg leq(y,z)) \\ \sigma &= \{x \leftarrow a,\ y \leftarrow w,\ z \leftarrow c\} \\ \phi\sigma &= \neg leq(w,a) \rightarrow \exists z(\neg leq(z,a) \wedge \neg leq(w,z)) \end{aligned}$$

Note

Substitution forbidden in cases such as:

ϕ **as above and** $\sigma = \{y \leftarrow f(z)\}$

Typed Signatures

Definition

A *typed Signature* is a tuple

$$\Sigma = (S, \leq, F, P, \alpha),$$

where

- S **is a finite set of *types* (or *sorts*)**
- $\leq$ **is a partial ordering on** S
- F, P **are sets of function and predicate symbols (as before)**
- $\alpha : F \cup P \longrightarrow S^*$ **assigns argument and domain types to function and predicate symbols**

Typed Signatures

The function α

$\alpha(f)$ = $Z_1 \ldots Z_n Z'$ **means:**

f **is a symbol for functions assigning to** n**-tuples of elements of type** $Z_1 \ldots Z_n$ **an element of type** Z'

$\alpha(p)$ = $Z_1, \ldots, Z_n$ **means:**

p **is a symbol for relations on** n**-tuples of elements of types** $Z_1, \ldots, Z_n$

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Variables are typed as well

For each type $Z \in S$ **there is an infinite set of variables of type** Z

Typed Signatures: Terms

- **If x is a variable of type Z, then x is a term of type Z**
- **If**
  - **$t_1, \ldots, t_n$ are terms of types $Y_1, \ldots, Y_n$**
  - **f is a functions symbol with $\alpha(f) = Z_1 \cdots Z_n Z'$**
  - **$Y_i \leq Z_i$ for all $1 \leq i \leq n$**

  then $f(t_1, \ldots, t_n)$ is a term of type Z'.

Typed Signatures: Formulas

- **If**

  – $t_1, \ldots, t_n$ **are terms with types** $Y_1, \ldots, Y_n$
  – p **is a predicate symbol** $\alpha(p) = Z_1 \cdots Z_n$
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- **If t, s are terms of sorts X and Y with $X \leq Y$ or $Y \leq X$, then $t \doteq s$ is a typed formula**

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- **If A, B are typed formulas, then so are**

$$\neg A \qquad (A \wedge B) \qquad (A \vee B) \qquad (A \rightarrow B)$$

- **If A is a typed formula and x is a typed variable, then**

$$\forall x A \qquad \exists x A$$

  are typed formulas

Typed Interpretations

Given a signature $\Sigma = (S, \leq, F, P, \alpha)$

Interpretation

A pair (D, I) **such that**

- $\{D_Z \mid Z \in S\}$ **is a family of non-empty sets with**
  - $D = \bigcup\{D_Z \mid Z \in S\}$
  - $D_{Z_1} \subseteq D_{Z_2}$ **if** $Z_1 \leq Z_2$
- $I(f) : D_{Z_1} \times \cdots \times D_{Z_n} \to D_{Z'}$ **if** $\alpha(f) = Z_1 \cdots Z_n Z'$
- $I(p) \subseteq D_{Z_1} \times \cdots \times D_{Z_n}$ **if** $\alpha(p) = Z_1 \cdots Z_n$

Typed Substitutions

Typed substitution

A substitution is well-sorted if for each variable x,

the type of the term $\sigma(x)$ is a sub-type of the type of x

Special Type Structures

A type structure $(S, \leq)$ is

- ***discrete*,**

  in case $Z_1 \leq Z_2$ only if $Z_1 = Z_2$

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- **a *tree structure*,**

  in case $U \leq Z_1$ and $U \leq Z_2$ implies $Z_2 \leq Z_1$ oder $Z_1 \leq Z_2$

- **a *lattice*,**

  in case that for any two sorts Z_1, Z_2 there is an infimum U, i.e.

  - $U \leq Z_1$ **and** $U \leq Z_2$
  - $W \leq U$ **for every sort** $W \in S$ **with** $W \leq Z_1$ **and** $W \leq Z_2$