
Formal Verification of Software

Propositional and Predicate Logic

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Propositional Logic: Syntax

Special symbols

() \neg \wedge \vee \rightarrow \leftrightarrow

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Signature

Propositional variables $\Sigma = \{p_0, p_1, \dots\}$

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Signature

Propositional variables $\Sigma = \{p_0, p_1, \dots\}$

Formulas

- The propositional variables $p \in \Sigma$ are formulas
- If A, B are formulas, then

$\neg A$ $(A \wedge B)$ $(A \vee B)$ $(A \rightarrow B)$ $(A \leftrightarrow B)$

are formulas

Propositional Logic: Unified Notation

Introduced by Smullyan, 1968

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Conjunctive formulas **Type** α

$$\neg\neg A \quad (A \wedge B) \quad \neg(A \vee B) \quad \neg(A \rightarrow B)$$

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Conjunctive formulas **Type α**

$$\neg\neg A \quad (A \wedge B) \quad \neg(A \vee B) \quad \neg(A \rightarrow B)$$

Disjunctive formulas **Type β**

$$\neg(A \wedge B) \quad (A \vee B) \quad (A \rightarrow B)$$

Propositional Logic: Unified Notation

Non-literal formulas and their corresponding “logical” sub-formulas

α	α_1	α_2
$A \wedge B$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \rightarrow B)$	A	$\neg B$
$\neg\neg A$	A	A

Propositional Logic: Unified Notation

Non-literal formulas and their corresponding “logical” sub-formulas

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B
$\neg\neg A$	A	A			

Propositional Logic: Semantics

Interpretation

Function $I : \Sigma \rightarrow \{\underline{\text{true}}, \underline{\text{false}}\}$

Propositional Logic: Semantics

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Function $I : \Sigma \rightarrow \{\underline{\text{true}}, \underline{\text{false}}\}$

Valuation

Extension of interpretation to formulas as follows:

$$\begin{aligned} \text{val}_I(p) &= I(p) \\ \text{val}_I(\neg p) &= \begin{cases} \underline{\text{true}} & \text{if } I(p) = \underline{\text{false}} \\ \underline{\text{false}} & \text{if } I(p) = \underline{\text{true}} \end{cases} \end{aligned}$$

Propositional Logic: Semantics

$$\text{val}_I(\alpha) = \begin{cases} \underline{\text{true}} & \text{if } \text{val}_I(\alpha_1) = \underline{\text{true}} \\ & \text{and } \text{val}_I(\alpha_2) = \underline{\text{true}} \\ \underline{\text{false}} & \text{if } \text{val}_I(\alpha_1) = \underline{\text{false}} \\ & \text{or } \text{val}_I(\alpha_2) = \underline{\text{false}} \end{cases}$$

Propositional Logic: Semantics

$$\begin{aligned} \text{val}_I(\alpha) &= \left\{ \begin{array}{l} \underline{\text{true}} \quad \text{if } \text{val}_I(\alpha_1) = \underline{\text{true}} \\ \quad \quad \quad \text{and } \text{val}_I(\alpha_2) = \underline{\text{true}} \\ \\ \underline{\text{false}} \quad \text{if } \text{val}_I(\alpha_1) = \underline{\text{false}} \\ \quad \quad \quad \text{or } \text{val}_I(\alpha_2) = \underline{\text{false}} \end{array} \right. \\ \\ \text{val}_I(\beta) &= \left\{ \begin{array}{l} \underline{\text{true}} \quad \text{if } \text{val}_I(\beta_1) = \underline{\text{true}} \\ \quad \quad \quad \text{or } \text{val}_I(\beta_2) = \underline{\text{true}} \\ \\ \underline{\text{false}} \quad \text{if } \text{val}_I(\beta_1) = \underline{\text{false}} \\ \quad \quad \quad \text{and } \text{val}_I(\beta_2) = \underline{\text{false}} \end{array} \right. \end{aligned}$$

Propositional Logic: Semantics

$$\mathbf{val}_I(A \leftrightarrow B) = \begin{cases} \mathbf{true} & \text{if } \mathbf{val}_I(A) = \mathbf{val}_I(B) \\ \mathbf{false} & \text{if } \mathbf{val}_I(A) \neq \mathbf{val}_I(B) \end{cases}$$

Predicate Logic: Syntax

Additional special symbols

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Object variables

Var = $\{x_0, x_1, \dots\}$

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Signature

Triple $\Sigma = \langle F_\Sigma, P_\Sigma, \alpha_\Sigma \rangle$ **consisting of**

- **set** F_Σ **of function symbols**
- **set** P_Σ **of predicate symbols**
- **function** $\alpha_\Sigma : F_\Sigma \cup P_\Sigma \rightarrow \mathbb{N}$
assigning arities to function and predicate symbols

Predicate Logic: Syntax

Terms

- **variables $x \in \text{Var}$ are terms**
- **if $f \in F_\Sigma$, $\alpha_\Sigma(f) = n$, and t_1, \dots, t_n terms, then $f(t_1, \dots, t_n)$ is a term**

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Atoms

If $p \in P_\Sigma$, $\alpha_\Sigma(p) = n$, and t_1, \dots, t_n terms, then $p(t_1, \dots, t_n)$ is an atom

Predicate Logic: Syntax

Formulas

- **Atoms are formulas**
- **If A, B are formulas, $x \in \text{Var}$, then**

$\neg A, (A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B), \forall x A, \exists x A$

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Literals

If A is an atom, then A and $\neg A$ are literals

Example

Signature

$$\Sigma_{\leq} = \langle \{0, a, b, f\}, \{in_iv, leq\}, \alpha \rangle$$

with

$$\alpha(0) = \alpha(a) = \alpha(b) = 0$$

$$\alpha(f) = 1$$

$$\alpha(leq) = 2$$

$$\alpha(in_iv) = 3 \quad \text{(in interval)}$$

Example

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Formula

$$\phi = \underbrace{\neg leq(y, x)}_{\text{Atom}} \rightarrow \underbrace{\exists z (\neg leq(z, x) \wedge \neg leq(y, z))}_{\text{Scope of } \exists z}$$

Predicate Logic: Unified Notation

Extension of unified notation for propositional logic

Universal formulas **Type** γ

$$\forall xA \quad \neg\exists xA$$

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Extension of unified notation for propositional logic

Universal formulas **Type γ**

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Existential formulas **Type δ**

$$\neg\forall xA \quad \exists xA$$

Predicate Logic: Unified Notation

γ - and δ -formulas and their corresponding “logical” sub-formulas

γ	$\gamma_1(x)$	δ	$\delta_1(x)$
$\forall x A(x)$	$A(x)$	$\neg \forall x A(x)$	$\neg A(x)$
$\neg \exists x A(x)$	$\neg A(x)$	$\exists x A(x)$	$A(x)$

Predicate Logic: Semantics

Interpretation

A pair $\mathcal{D} = \langle D, I \rangle$ where

- D an arbitrary non-empty set, the *universe*
- I an interpretation function

for $f \in F_\Sigma$: $I(f) : D^{\alpha(f)} \rightarrow D$

for $p \in P_\Sigma$: $I(p) : D^{\alpha(p)} \rightarrow \{\underline{\text{true}}, \underline{\text{false}}\}$

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Variable assignment

A function $\lambda : \text{Var} \rightarrow D$

Predicate Logic: Semantics

Valuation

Extension of interpretation and variable assignment to formulas

$$\mathbf{val}_{\mathcal{D},\lambda}(x) = \lambda(x) \quad \mathbf{for} \ x \in \mathbf{Var}$$

$$\mathbf{val}_{\mathcal{D},\lambda}(f(t_1, \dots, t_n)) = I(f)(\mathbf{val}_{\mathcal{D},\lambda}(t_1), \dots, \mathbf{val}_{\mathcal{D},\lambda}(t_n))$$

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$$\mathbf{val}_{\mathcal{D},\lambda}(p(t_1, \dots, t_n)) = I(p)(\mathbf{val}_{\mathcal{D},\lambda}(t_1), \dots, \mathbf{val}_{\mathcal{D},\lambda}(t_n))$$

$$\mathbf{val}_{\mathcal{D},\lambda}(\forall x A) = \begin{cases} \underline{\mathbf{true}} & \mathbf{if } \mathbf{val}_{\mathcal{D},\lambda_x^d}(A) = \underline{\mathbf{true}} \quad \mathbf{for all } d \in D \\ \underline{\mathbf{false}} & \mathbf{otherwise} \end{cases}$$

$$\mathbf{val}_{\mathcal{D},\lambda}(\exists x A) = \begin{cases} \underline{\mathbf{true}} & \mathbf{if } \mathbf{val}_{\mathcal{D},\lambda_x^d}(A) = \underline{\mathbf{true}} \quad \mathbf{for some } d \in D \\ \underline{\mathbf{false}} & \mathbf{otherwise} \end{cases}$$

$\mathbf{val}_{\mathcal{D},\lambda}$ defined for propositional operators in the same way as \mathbf{val}_I .

Predicate Logic: Semantics

Example

$$D = \mathbb{R}$$

$$I(0) = 0$$

$$I(a) = -1$$

$$I(b) = 1$$

$$I(f) = \begin{cases} \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto x^2 \end{cases}$$

$$I(\text{leq}) = \underline{\text{true}} \quad \text{iff} \quad x \leq_{\mathbb{R}} y$$

$$I(\text{in_iv}) = \underline{\text{true}} \quad \text{iff} \quad x \in [a, b]$$

Predicate Logic: Semantics

Model

An interpretation \mathcal{D} is model of a set Φ of formulas iff

$\text{val}_{\mathcal{D},\lambda}(A) = \underline{\text{true}}$ for all λ and all $A \in \Phi$.

Notation: $\mathcal{D} \models \Phi$

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Satisfiable

Φ is satisfiable iff there are

an interpretation \mathcal{D} and a variable assignment λ s.t.

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Validity

A is valid iff

all interpretations are a model of A

Predicate Logic: Semantics

Consequence

**A formula A is a consequence of Φ iff
all models of Φ are models of A as well**

Notation: $\Phi \models A$

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Equivalent formulas

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Satisfiability equivalent formulas

Two formulas are satisfiability equivalent iff
they are either both satisfiable or both unsatisfiable

Substitutions

Substitution

Function $\sigma : \mathbf{Var} \rightarrow \mathbf{Term}$

Written as: $\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$

where $\sigma(x) = \begin{cases} t_i & \mathbf{if } x = x_i \mathbf{ for } 1 \leq i \leq n \\ x & \mathbf{otherwise} \end{cases}$

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Extension to terms and formulas

By replacing all *free* occurrences of variables x by $\sigma(x)$

Substitutions

Example:

$$\phi = \neg leq(y, x) \rightarrow \exists z(\neg leq(z, x) \wedge \neg leq(y, z))$$

$$\sigma = \{x \leftarrow a, y \leftarrow w, z \leftarrow c\}$$

$$\phi\sigma = \neg leq(w, a) \rightarrow \exists z(\neg leq(z, a) \wedge \neg leq(w, z))$$

Substitutions

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Note

Substitution forbidden in cases such as:

ϕ as above and $\sigma = \{y \leftarrow f(z)\}$

Typed Signatures

Definition

A *typed Signature* is a tuple

$$\Sigma = (S, \leq, F, P, \alpha),$$

where

- S is a finite set of *types* (or *sorts*)
- \leq is a partial ordering on S
- F, P are sets of function and predicate symbols (as before)
- $\alpha : F \cup P \rightarrow S^*$ assigns argument and domain types to function and predicate symbols

Typed Signatures

The function α

$\alpha(f) = Z_1 \dots Z_n Z'$ means:

f is a symbol for functions assigning to n -tuples of elements of type $Z_1 \dots Z_n$ an element of type Z'

$\alpha(p) = Z_1, \dots, Z_n$ means:

p is a symbol for relations on n -tuples of elements of types Z_1, \dots, Z_n

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Variables are typed as well

For each type $Z \in S$ there is an infinite set of variables of type Z

Typed Signatures: Terms

• **If x is a variable of type Z , then x is a term of type Z**

• **If**

– t_1, \dots, t_n are terms of types Y_1, \dots, Y_n

– f is a functions symbol with $\alpha(f) = Z_1 \cdots Z_n Z'$

– $Y_i \leq Z_i$ for all $1 \leq i \leq n$

then $f(t_1, \dots, t_n)$ is a term of type Z' .

Typed Signatures: Formulas

• **If**

- t_1, \dots, t_n are terms with types Y_1, \dots, Y_n
- p is a predicate symbol $\alpha(p) = Z_1 \cdots Z_n$
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- If t, s are terms of sorts X and Y with $X \leq Y$ or $Y \leq X$, then $t \doteq s$ is a typed formula

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- **If A, B are typed formulas, then so are**

$$\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \rightarrow B)$$

- **If A is a typed formula and x is a typed variable, then**

$$\forall x A \quad \exists x A$$

are typed formulas

Typed Interpretations

Given a signature $\Sigma = (S, \leq, F, P, \alpha)$

Interpretation

A pair (D, I) such that

- $\{D_Z \mid Z \in S\}$ is a family of non-empty sets with
 - $D = \bigcup \{D_Z \mid Z \in S\}$
 - $D_{Z_1} \subseteq D_{Z_2}$ **if** $Z_1 \leq Z_2$
- $I(f) : D_{Z_1} \times \cdots \times D_{Z_n} \rightarrow D_{Z'}$ **if** $\alpha(f) = Z_1 \cdots Z_n Z'$
- $I(p) \subseteq D_{Z_1} \times \cdots \times D_{Z_n}$ **if** $\alpha(p) = Z_1 \cdots Z_n$

Typed Substitutions

Typed substitution

**A substitution is well-sorted if for each variable x ,
the type of the term $\sigma(x)$ is a sub-type of the type of x**

Special Type Structures

A type structure (S, \leq) is

• *discrete*,

in case $Z_1 \leq Z_2$ only if $Z_1 = Z_2$

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- ***a tree structure,***

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- ***a tree structure,***

in case $U \leq Z_1$ and $U \leq Z_2$ implies $Z_2 \leq Z_1$ oder $Z_1 \leq Z_2$

- ***a lattice,***

in case that for any two sorts Z_1, Z_2 there is an infimum U , i.e.

- **$U \leq Z_1$ and $U \leq Z_2$**

- **$W \leq U$ for every sort $W \in S$ with $W \leq Z_1$ and $W \leq Z_2$**