CADE Tutorial
The Sequent Calculus of the KeY Tool
Part II

Reiner Hähnle & Peter H. Schmitt

Department of Computer Science
Technische Universität Darmstadt / Karlsruhe Institute of Technology

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Typed Logic

Modular Reasoning

Extension of First-Order Logic

Undefinedness

Theories

Wrap-Up

Typed First-Order Logic

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- a set of types $\mathcal{T}$ and a subtype relation $\sqsubseteq$ on $\mathcal{T} \times \mathcal{T}$.

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- Type spec $f : T_1 \times \ldots T_n \to T_0$ required for $f \in \mathit{Func}$,
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is a term of type T_0.

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Definition of Formulas Unchanged

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Transform into untyped predicate logic

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Extend Calculus

- A Calculus for Type Predicates and Type Coercion
  Martin Giese
  TABLEAUX 2005, SLNCS Vol.3702, pp 123-137
- A Calculus for Typed First-Order Logic
  Peter H. Schmitt and Mattias Ulbrich
  FM 2015

First-Order Rules

$$\text{allRight}\ \frac{\Gamma \Longrightarrow [x/c](\varphi), \Delta}{\Gamma \Longrightarrow (\forall\, A\, x)\varphi, \Delta}$$
$c : \to A$ a new constant

$$\text{allLeft}\ \frac{\Gamma, \forall\, A\, x)\varphi, [x/t](\varphi) \Longrightarrow \Delta}{\Gamma, (\forall\, A\, x)\varphi \Longrightarrow \Delta}$$
t ground term, $\sigma(t) \sqsubseteq A$

$$\text{exLeft}\ \frac{\Gamma, [x/c](\varphi) \Longrightarrow \Delta}{\Gamma, (\exists\, A\, x).\varphi \Longrightarrow \Delta}$$
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$$\text{exRight}\ \frac{\Gamma \Longrightarrow (\exists\, A\, x)\varphi, [x/t](\varphi), \Delta}{\Gamma \Longrightarrow (\exists\, A\, x)\varphi, \Delta}$$
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$$\text{close}\ \frac{}{\Gamma, \varphi \Longrightarrow \varphi, \Delta}$$

$$\text{closeFalse}\ \frac{}{\Gamma, \mathrm{false} \Longrightarrow \Delta}$$

$$\text{closeTrue}\ \frac{}{\Gamma \Longrightarrow \mathrm{true}, \Delta}$$

Equational Rules

$$\text{eqLeft}\ \frac{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi), [z/t_2](\varphi) \Longrightarrow \Delta}{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi) \Longrightarrow \Delta}$$
$$\text{if } \sigma(t_2) \sqsubseteq \sigma(t_1)$$

$$\text{eqRight}\ \frac{\Gamma, t_1 \doteq t_2 \Longrightarrow [z/t_2](\varphi), [z/t_1](\varphi), \Delta}{\Gamma, t_1 \doteq t_2 \Longrightarrow [z/t_1](\varphi), \Delta}$$
$$\text{if } \sigma(t_2) \sqsubseteq \sigma(t_1)$$

$$\text{eqSymmLeft}\ \frac{\Gamma, t_2 \doteq t_1 \Longrightarrow \Delta}{\Gamma, t_1 \doteq t_2 \Longrightarrow \Delta} \qquad \text{eqReflLeft}\ \frac{\Gamma, t \doteq t \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta}$$

A Closer Look at eqLeft

$$\text{eqLeft}\ \frac{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi), [z/t_2](\varphi) \Longrightarrow \Delta}{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi) \Longrightarrow \Delta}$$

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Why is the side condition $\sigma(t_2) \sqsubseteq \sigma(t_1)$ necessary?

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Consider the signature:

$B \sqsubsetneq A$, $a : \ \to A$, $b : \ \to B$, $p : B$.

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Why is the side condition $\sigma(t_2) \sqsubseteq \sigma(t_1)$ necessary?
Consider the signature:
$B \sqsubsetneqq A$, $a : \ \to A$, $b : \ \to B$, $p : B$.
Applying eqLeft without side condition on the sequent

$$b \doteq a, p(b) \Longrightarrow$$

would result in

$$b \doteq a, p(b), p(a) \Longrightarrow$$

with $p(a)$ being not well-typed.

Soundness and Completeness

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presents a sound and complete calculus
provided the type hierarchy $\mathcal{H}$ is closed under greatest lower bounds.

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Optimization

A sound and complete calculus without restrictions on $\mathcal{H}$
can be obtained by the following modification:

$$\text{eqLeft}\ \frac{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi), [z/t_2](\varphi) \Longrightarrow \Delta}{\Gamma, t_1 \doteq t_2, [z/t_1](\varphi) \Longrightarrow \Delta}$$
$$\text{provided } [z/t_2](\varphi) \text{ is welltyped}$$

Likewise for eqRight.

Modular Reasoning

Theoretical Motivation

$\neg(\exists x)(\exists y)(x \doteq y)$
is tautology
for this type hierarchy

$$
\begin{tikzcd}
 & \top & \\
A \arrow[ur] & & B \arrow[ul] \\
 & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$A : x$
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\end{tikzcd}$$

The phenomenon that universal validity of a formula depends on symbols not occurring in it, is highly undesirable.

Modular Reasoning

Practical Motivation from Program Verification

Segment of Java Library

$$
\begin{tikzcd}
 & \text{java.lang.Object} & \\
 & \text{java.util.AbstractCollection} \arrow[u] & \\
 & \text{java.util.AbstractList} \arrow[u] & \\
\text{java.util.ArrayList} \arrow[ur] & \text{AbstractSequentialList} \arrow[u] & \text{Vector} \arrow[ul]
\end{tikzcd}
$$

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Practical Motivation from Program Verification

Segment of Java Library

java.lang.Object
↑
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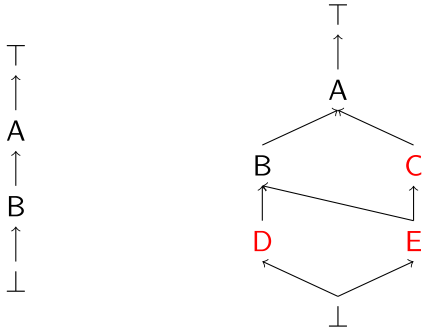
↗ ↑ ↖
java.util.ArrayList AbstractSequentialList Vector

Verification of programs using these classes should remain valid if another subclass is added to java.util.AbstractList.

Type Hierarchies Extensions

$$(\mathrm{TSym}_1, \sqsubseteq_1) \quad \sqsubseteq \quad (\mathrm{TSym}_2, \sqsubseteq_2)$$



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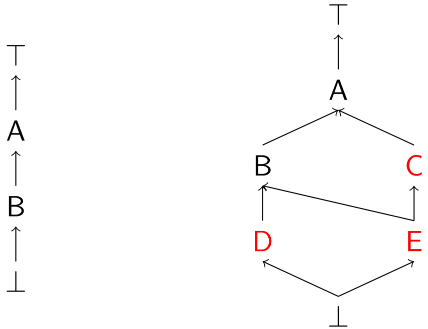

Only subtype relations may be added.

Logical Consequence Relation

Definition of Super-Consequence Relation

$\varphi \in \mathrm{Fml}_{\mathcal{T},\Sigma}$, $\Phi \subseteq \mathrm{Fml}_{\mathcal{T},\Sigma}$
for type hierarchy $\mathcal{T}$ and signature Σ.

$\Phi \vdash \varphi$ iff for all type hierarchies $\mathcal{T}'$ with $\mathcal{T} \sqsubseteq \mathcal{T}'$
and all $\mathcal{T}' - \Sigma$-structures $\mathcal{M}$
if $\mathcal{M} \models \Phi$ then $\mathcal{M} \models \varphi$

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For the example on the previous slide.

$$\emptyset \not\vdash \neg(\exists x)(\exists y)(x \doteq y)$$

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Note

Completeness

A calculus that is complete for the ordinary consequence relation is also complete for the super-consequence relation.

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Soundness

A rule that is sound for the ordinary consequence relation
need not be sound for the super-consequence relation.

Conditional Terms

Syntax

(if φ then t_1 else t_2) is a term of type A
for a formula φ and terms t_i of type A_i
if $A_2 \sqsubseteq A_1 = A$ or $A_1 \sqsubseteq A_2 = A$.

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Semantics by Elimination (Example)

$(\forall x, y)(x \leqslant (\text{if } x \leqslant y \text{ then } y \text{ else } x))$
is replaced by
$(\forall x, y)(x \leqslant y \to x \leqslant y \land \neg(x \leqslant y) \to y \leqslant y)$

Motivation

At the VSTTE'10 (Verified Software:Theories, Tools and Experiments) conference 2010 in Edinburgh.

VSComp: The Verified Software Competition

Problem 1

- Description: Given an N-element array of natural numbers, write a program to compute the sum and the maximum of the elements in the array.
- Properties: Given that $N \geqslant 0$ and $a[i] \geqslant 0$ for $0 \leqslant i < N$, prove the post-condition that $sum \leqslant N * max$

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We need a way to express $\Sigma_{i=b}^{i=e}$.

Variable Binders

Explanation

Variable Binders are function symbols which bind a variable ranging over a set of values.

Application of a variable binder results in a term.

Variable Binders available in KeY

Mathematical Notation	KeY Syntax
$\sum_{b_0 \leqslant vi < b_1} s_1$	$bsum\{Int\ vi;\}(b_0, b_1, s_1)$
$\prod_{b_0 \leqslant vi < b_1} s_1$	$prod\{Int\ vi;\}(b_0, b_1, s_1)$
$\bigcup_{-\infty < vi < \infty} s_2$	$infiniteUnion\{Int\ vi;\}(s_2)$
$\langle s_3[b_0/vi], \ldots, s_3[(b_1 - 1)/vi]\rangle$	$seqDef\{Int\ vi;\}(b_0, b_1, s_3)$

b_0, b_1, s_1 terms of type *Int*,
s_2 term of type *LocSet*,
s_3 term of type *Seq*,
vi would typically occur free in s_1, s_2, s_3

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Theorem

For every formula containing variable binders there is a satisfiability equivalent formula without.

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Example

$(\forall\, Int\ N)(bsum\{Int\ vi;\}(0, N, a[vi]) \leqslant N * \ max)$

is replaced by

$$
\begin{array}{l}
f(0) = 0 \quad \wedge \\
(\forall\, Int\, j)(f(j+1) = f(j) + a[j]) \quad \wedge \\
(\forall\, Int\ N)(f(N) \leqslant N * \ max)
\end{array}
$$

Ways to deal with undefinedness

1. Remove it.
   e.g. set the empty sum to 0: $\Sigma_{i=0}^{i=-2} s_i = 0$.
2. Introduce new error elements.
3. Use some kind of 3-valued logic.
4. Use underspecification.

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   $\frac{1}{0} \doteq \frac{2}{0}$ is not.

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   $(\exists\, i)(\frac{1}{0} \doteq i)$ is a tautology.

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   Also $cast_{Int}(c) \doteq 5 \rightarrow c \doteq 5$ is not a tautology. In case c is not of type *Int* the underspecified value for $cast_{Int}(c)$ could be 5.

Axioms for $\mathbb{Z}$

$A.1$	$(i + j) + k \doteq i + (j + k)$
$A.2$	$i + j \doteq j + i$
$A.3$	$0 + i \doteq i$
$A.4$	$i + (-i) \doteq 0$
$M.1$	$(i * j) * k \doteq i * (j * k)$
$M.2$	$i * j \doteq j * i$
$M.3$	$1 * x \doteq x$
$M.4$	$i * (j + k) \doteq i * j + i * k$
$M.5$	$1 \neq 0$
$O.1$	$0 < i \vee 0 = i \vee 0 < (-i)$
$O.2$	$0 < i \wedge 0 < j \rightarrow 0 < i + j$
$O.3$	$0 < i \wedge 0 < j \rightarrow 0 < i * j$
$O.4$	$i < j \leftrightarrow 0 < j + (-i)$
$O.5$	$\neg(0 < 0)$
Ind	$\varphi(0) \wedge (\forall i)(0 \leqslant i \wedge \varphi(i) \rightarrow \varphi(i + 1)) \rightarrow (\forall i)(0 \leqslant i \rightarrow \varphi(i))$

Transitive Closure

Definition

Let (G, R) be a graph and G a type.

$$reachR(g, h, 0) \leftrightarrow g \doteq h)$$
$$n \geqslant 0 \rightarrow (reachR(g, h, n + 1) \leftrightarrow (\exists\, G\, k)(reachR(g, k, n) \wedge R(k, h)))$$
$$TR(g, h) \leftrightarrow (\exists\, Int\, n)(n \geqslant 0 \wedge reachR(g, h, n))$$

Here, g, h are variables of type G and n of type Int.

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TR is the transitive closure of R.
More precisely:
In any interpretation satisfying the above three axioms the interpretation of TR is the transitive closure of R.

Proof Exercise

A Special Case of the Bellman-Ford Lemma

Let (G, R) be a graph, $s \in G$ the start element.
Let $d : G \to \mathbb{N}$ be a function satisfying:

$$
\begin{array}{l}
d(s) = 0 \\
(\forall\, G\, g)(g \neq s \to (\exists\, G\, h)(R(h, g) \land d(g) = d(h) + 1)) \\
(\forall\, G\, g)(\forall G\ \ h; (R(h, g) \to d(g) <= d(h) + 1))
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Then $d(g)$ is the length of the shortest path from s to g.

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KeY Demo

A Note on Completeness

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The axioms on $\mathbb{Z}$ shown previously fall short of being complete.

Goodstein's theorem

$$(\forall\, Int\, m)(m > 0 \rightarrow (\exists\, Int\, n)(n > 1 \wedge G(m)(n) \doteq 0$$

cannot be derived.
$G(m)(1), G(m)(2), \ldots, G(m)(n), \ldots$ is the Goodstein sequence of m.

The Data Type of Finite Sequences

Core Theory

$seqLen : Seq \to Int$

$seqGet_A : Seq \times Int \to A$ for any type $A \sqsubseteq Any$

$seqGetOutside : Any$

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Definitional Extension

$seqEmpty : Seq$

$seqSingleton : Any \to Seq$

$seqConcat : Seq \times Seq \to Seq$

$seqSub : Seq \times Int \times Int \to Seq$

$seqReverse : Seq \to Seq$

$seqIndexOf : Seq \times Any \to Int$

$seqNPerm(Seq)$

$seqPerm(Seq, Seq)$

$seqSwap : Seq \times Int \times Int \to Seq$

$seqRemove : Seq \times Int \to Seq$

$seqNPermInv : Seq \to Seq$

$seqDepth : Seq \to Int$

Sequence Comprehension

Let le, ri be integers, t a term of arbitrary type, typically containing variable vi:

$$seqDef\{Int\ vi;\}(le, ri, t)$$

is interpreted as the sequence

$$\langle t(le/vi), \ldots, t((ri-1)/vi)\rangle$$

If $ri \leqslant le$ it is the empty sequence.

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If $ri \leqslant le$ it is the empty sequence.

Example

$$seqDef\{Int\ vi;\}(2, 6, vi^2)$$

is interpreted as the sequence

$$\langle 4, 9, 25 \rangle$$

Axioms of the core theory CoT_{seq}

1 $(\forall\, Seq\, s)(0 \leqslant seqLen(s))$

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$(\forall\, Int\; i)(0 \leqslant i < seqLen(s_1) \rightarrow seqGet_{Any}(s_1, i) \doteq seqGet_{Any}(s_2, i)))$

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3 $(\forall\, Int\, ri, le)($
$(le < ri \rightarrow seqLen((seqDef\{u\}(le, ri, t)) \doteq ri - li)$
$\wedge$
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   $\wedge$
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4. $(\forall\, Int\, i, ri, le)(\forall\, Any\, \bar{x})(((0 \leqslant i \wedge i < ri - le) \rightarrow$
   $seqGet_A(seqDef\{u\}(le, ri, t), i) \doteq cast_A(t\{(le + i)/u\}))$
   $\wedge$
   $(\neg(0 \leqslant i \wedge i < ri - le) \rightarrow$
   $seqGet_A(seqDef\{u\}(le, ri, t), i) \doteq cast_A(seqGetOutside)))$

A Model for CoT_{seq}

The type domain D^{Seq}

Inductive Definition:

$$
\begin{array}{lcl}
U & = & D^{Any} \setminus D^{Seq} \\
D^0_{\mathrm{Seq}} & = & \{\langle\rangle\} \\
D^{n+1}_{\mathrm{Seq}} & = & \{\langle a_1, \ldots, a_k \rangle \mid k \in \mathbb{N} \text{ and } a_i \in D^n_{Seq} \cup U, 1 \leqslant i \leqslant k\}, n \geqslant 0
\end{array}
$$

$$
D^{Seq} \quad := \quad \bigcup_{n \geqslant 0} D^n_{\mathrm{Seq}}
$$

A Model for CoT_{seq}

Interpretaion of the Vocabulary of CoT_{seq}

1. $seqGet_A^{\mathcal{M}}(\langle a_0, \ldots, a_{n-1}\rangle, i) =$
$$\begin{cases} cast_A^{\mathcal{M}}(a_i) & \text{if } 0 \leqslant i < n \\ cast_A^{\mathcal{M}}(seqGetOutside^{\mathcal{M}}) & \text{otherwise} \end{cases}$$
2. $seqLen^{\mathcal{M}}(\langle a_0, \ldots, a_{n-1}\rangle) = n$
3. $seqGetOutside^{\mathcal{M}} \in D^{Any}$ arbitrary.
4. $seqDef\{iv\}(le, ri, e)^{\mathcal{M},\beta} =$
$$\begin{cases} \langle a_0, \ldots a_{k-1}\rangle & \text{if } ri - le = k > 0 \text{ and for all } 0 \leqslant i < k \\ & a_i = e^{\mathcal{M},\beta_i} \text{ with } \beta_i = \beta[le + i/iv] \\ \langle\rangle & \text{otherwise} \end{cases}$$

Consistency

Theorem

The theory CoT_{seq} is consistent.

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The theory CoT_{seq} is consistent.

Proof

Check that all axioms of CoT_{seq} are true in the model $\mathcal{M}$.

Definitional Extension T_{seq}

Set 1

$$seqEmpty \doteq seqDef\{iv\}(0, 0, x)$$

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$$\begin{array}{l}(\forall\, Seq\; s_1, s_2)(seqConcat(s_1, s_2) \doteq \\ \quad seqDef\{iv\}(0, Len(s_1) + Len(s_2), \begin{array}[t]{l}\text{if } iv < Len(s_1) \\ \text{then } seqGet_{Any}(s_1, iv) \\ \text{else } seqGet_{Any}(s_2, iv - Len(s_1))\end{array}))\end{array}$$

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$$(\forall\, Seq\; s)(\forall\, Int\; i, j)(seqSub(s, i, j) \doteq seqDef\{iv\}(i, j, seqGet_{Any}(s, iv)))$$

Definitional Extension T_{seq}

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$$(\forall\, Seq\, s)(\forall\, Int\, i, j)(seqSub(s, i, j) \doteq seqDef\{iv\}(i, j, seqGet_{Any}(s, iv)))$$

$$\begin{array}{l}(\forall\, Seq\, s)(seqReverse(s) \doteq \\ \quad seqDef\{iv\}(0, Len(s), seqGet_{Any}(s, Len(s) - iv - 1)))\end{array}$$

For concise presentation we have used *Len* instead of *seqLen*.

Definitional Extension T_{seq}

Set 2

$$(\forall\, Seq\, s)(seqNPerm(s) \leftrightarrow$$
$$(\forall\, Int\, i)(0 \leqslant i < Len(s) \rightarrow (\exists\, Int\, j)(0 \leqslant j < Len(s) \wedge seqGet_{Int}(s, j) \doteq i)))$$

$$(\forall\, Seq\, s_1, s_2)(seqPerm(s_1, s_2) \leftrightarrow Len(s_1) \doteq Len(s_2) \wedge$$
$$(\exists\, Seq\, s)(Len(s) \doteq Len(s_1) \wedge seqNPerm(s) \wedge$$
$$(\forall\, Int\, i)(0 \leqslant i < Len(s) \rightarrow$$
$$seqGet_{Any}(s_1, i) \doteq seqGet_{Any}(s_2, seqGet_{Int}(s, i))))))))))$$

For concise presentation we have again used *Len* instead of *seqLen*.

Consistency

Theorem

The theory T_{seq} is consistent.

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Theorem

The theory T_{seq} is consistent.

Proof

We refer to

If T_2 is obtained from T_1 by definitional extension then T_2 is consistent iff T_1 is consistent.

and the fact that T_{seq} is a definitional extension of CoT_{seq}.

Relative Completeness

Theorem

If the union of the component theories is complete then

$$T_{seq} \text{ is complete}$$

provided that the function symbol and axioms for *seqDepth* are added.

$$\begin{aligned} seqDepth(s) &= 0 \quad \text{if } \neg instance_{\mathrm{Seq}}(s) \\ seqDepth(s) &= max\{seqDepth(seqGet_{Seq}(s,i)) \mid \\ &\quad 0 \leqslant i < seqLen(s) \wedge instance_{\mathrm{Seq}}(seqGet_{Seq}(s,i))\} + 1 \end{aligned}$$

Derived Theorems in T_{seq}

1 getOfSeqConcat

$$(\forall\, Seq\; s, s2)(\forall\, Int\; i)(seqGet_{alpha}(seqConcat(s, s2), i) \doteq$$
$$\text{if } \; i < Len(s) \text{ then } \; seqGet_{alpha}(s, i) \text{ else } \; seqGet_{alpha}(s2, i - Len(s))$$

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KeY Demo

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2 getOfSeqSub
$(\forall\, Seq\; s)(\forall\, Int\; f, t, i)(seqGet_{alpha}(seqSub(s, f, t), i) \doteq$
$\text{if } 0 \leqslant i \wedge i < (t - f) \text{ then } \; seqGet_{alpha}(s, i + f) \text{ else } \; (alpha)seqGetOutside$

Derived Theorems in T_{seq}

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$(\forall\, Seq\; s, s2)(Len(seqConcat(s, s2)) \doteq Len(s) + Len(s2))$

Derived Theorems in T_{seq}

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   $(\forall\, Seq\, s, s2)(\forall\, Int\, i)(seqGet_{alpha}(seqConcat(s, s2), i) \doteq$
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3. lenOfSeqConcat
   $(\forall\, Seq\, s, s2)(Len(seqConcat(s, s2)) \doteq Len(s) + Len(s2))$
4. lenOfSeqSub
   $(\forall\, Seq\, s)(\forall\, Int\, from, to)(Len(seqSub(s, from, to)) \doteq$
   $\text{if } \ from < to \ \text{ then } \ (to - from) \ \text{ else } \ 0)$

Typed Logic

Modular Reasoning

Extension of First-Order Logic

Undefinedness

Theories

Wrap-Up

Requirements on the KeY Calculus, Re-Revisited

for serious program verification

- Full typed first-order logic
- Partially ordered extensible type hierarchies reflecting Java's type system
- Rich expressive power, even if theoretically redundant
- Coverage of partial functions
- Combination of automatic and interactive proving
- Extensible: many theories
- Supportive user interface

THE END

Derived Theorems on Permutations

seqNPermRange

$$(\forall\, Seq\, s)(\forall\, Int\, i)(seqNPerm(s) \wedge 0 \leqslant i \wedge i < Len(s))$$
$$\quad \rightarrow (0 \leqslant seqGet_{\mathrm{Int}}(s,i) \wedge seqGet_{\mathrm{Int}}(s,i) < Len(s))$$

seqNPermInjective

$$(\forall\, Seq\, s)(\forall\, Int\, i,j)(seqNPerm(s) \wedge 0 \leqslant i \wedge i < Len(s) \wedge$$
$$seqGet_{\mathrm{Int}}(s,i) \doteq seqGet_{\mathrm{Int}}(s,j)) \wedge 0 \leqslant j \wedge j < Len(s)$$
$$\quad \rightarrow i \doteq j)$$

seqNPermComp

$$(\forall\, Seq\, s1, s2)(seqNPerm(s1) \wedge seqNPerm(s2) \wedge Len(s1) \doteq Len(s2) \rightarrow$$
$$seqNPerm(seqDef\{u\}(0, Len(s1), seqGet_{\mathrm{Int}}(s1, seqGet_{\mathrm{Int}}(s2,u))))))$$

seqPermTrans

$$(\forall\, Seq\, s1, s2, s3)(seqPerm(s1,s2) \wedge seqPerm(s2,s3)$$
$$\quad \rightarrow seqPerm(s1,s3))$$

seqPermRefl $\quad (\forall\, Seq\, s)(seqPerm(s,s))$

Hereditary base-n notation
Taken from Wikipedia

The hereditary base-n notation for a natural number m is obtained from its ordinary base-n notation

$$m = m_k \cdot n^k + m_{k-1} \cdot n^{k-1} + \ldots m_1 \cdot n + m_0, \quad 0 \leqslant m_i < n, m_k \neq 0$$

by also writing the exponents k, $k-1$, ..., $n+1$ in base-n notation and again the thus arising exponents, and so on.

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by also writing the exponents k, $k-1$, ..., $n+1$ in base-n notation and again the thus arising exponents, and so on.

Example

base-2	35	$=$	$2^5 + 2^1 + 2^0$
hereditary base-2	35	$=$	$2^{2^2+1} + 2 + 1$
base-3	100	$=$	$3^4 + 2 \cdot 3^2 + 3^0$
hereditary base-3	100	$=$	$3^{3+1} + 2 \cdot 3^2 + 1.$

Goodstein Sequences

Taken from Wikipedia

Definition

$G(m)(1) = m$	
$G(m)(n+1)$	write $G(m)(n)$ in hereditary base-$(n+1)$ notation
	replace all bases $n+1$ by $n+2$
	subtract 1

Goodstein Sequences

Taken from Wikipedia

Definition

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\begin{array}{ll}
G(m)(1) = m & \\
G(m)(n+1) & \text{write } G(m)(n) \text{ in hereditary base-}(n+1) \text{ notation} \\
 & \text{replace all bases } n+1 \text{ by } n+2 \\
 & \text{subtract } 1
\end{array}
$$

Example

Base	Hered. not.	$G(3)(n)$	Notes
2	$2^1 + 1$	3	Write 3 in base 2 notation
3	$3^1 + 1 - 1 = 3^1$	3	Switch 2 to 3, subtract 1
4	$4^1 - 1 = 3$	3	Switch 3 to 4, subtract 1.No 4s left
5	$3 - 1 = 2$	2	No 4s left. Just subtract 1
6	$2 - 1 = 1$	1	No 5s left. Just subtract 1
7	$1 - 1 = 0$	0	No 6s left. Just subtract 1