An Improved Rule for While Loops in Deductive Program Verification

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Traditional Invariant Rule

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2. *Inv* is indeed an invariant

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Usually we also have a Δ there which we omit here. Can be negated and put into Γ.

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- Java programming language (abrupt termination)
  - break, (continue)
  - exceptions
  - return

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$$\frac{}{x \doteq 0 \not\vdash [\texttt{while } x \leq 5 \texttt{ do } x = x + 1;\ \texttt{od}]x \doteq 0}$$

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$$\frac{x \doteq 0 \vdash \mathit{Inv} \qquad x \doteq 0,\ \mathit{Inv},\ x \leq 5 \vdash [x = x + 1]\mathit{Inv} \qquad x \doteq 0,\ \mathit{Inv},\ \neg x \leq 5 \vdash x \doteq 0}{x \doteq 0 \nvdash [\texttt{while } x \leq 5 \texttt{ do } x = x + 1;\ \texttt{od}]x \doteq 0}$$

With $\mathit{Inv} \equiv \mathit{true}$ all premisses are valid but the conlusion is not.

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Anonymous update $\mathcal{V}$ that assigns fixed, unknown values to all locations.

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Can be written as taclet!

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$$\frac{\begin{array}{l} x \doteq 0 \vdash Inv \\ x \doteq 0,\ \{x := c\}Inv,\ \{x := c\}x \leq 5 \vdash \{x := c\}[x = x + 1]Inv \\ x \doteq 0,\ \{x := c\}Inv,\ \{x := c\}\neg x \leq 5 \vdash \{x := c\}x \doteq 0 \end{array}}{x \doteq 0 \not\vdash [\texttt{while}\ x \leq 5\ \texttt{do}\ x = x + 1;\ \texttt{od}]x \doteq 0}$$

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In fact we do not enumerate all locations and assign unknown values to them. Rather, we really use a special update. The update simplifier knows how to handle this special update, i.e. everything to the left of the special update must not be used for update simplification. This, in facts, is similar to throwing away the context—but can be expressed as a taclet.

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$$\frac{\Gamma \vdash \mathcal{U}\mathit{Inv} \qquad \mathit{Inv},\ \texttt{exp} \vdash \overbrace{[\ldots\texttt{break;}\ldots]\mathit{Inv}}^{\equiv \mathit{true}} \qquad \mathit{Inv},\ \neg\texttt{exp} \vdash \varphi}{\Gamma \vdash \mathcal{U}[\texttt{while (exp) \{...break;...\}}]\varphi}$$

2nd premiss trivially valid in case of abrupt termination!

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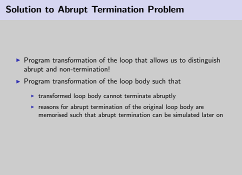
- Program transformation of the loop that allows us to distinguish abrupt and non-termination!
- Program transformation of the loop body such that
  - transformed loop body cannot terminate abruptly
  - reasons for abrupt termination of the original loop body are memorised such that abrupt termination can be simulated later on



Instead of such a transformation one could also introduce new modalities to distinguish abstract and non-termination. But this has 2 major drawbacks:

- Less efficient since the calculus would have to execute the loop body twice: first within a normal box and second within the new modality
- lots of new calculus rule required for the additional modalities

An Example

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while (i<100) {
   if (i==3)
     continue;
   j=j/i;
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⇝

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boolean cont=false;
boolean exc=false;
java.lang.Throwable theExc;
try {
  body: {
    if (i<100) {
      if (i==3) {
          cont=true;
          break body;
      }
      j=j/i;
      i++;
    }
  }
} catch (java.lang.Throwable e) {
    exc=true;
    theExc=e;
}
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Rule Respecting Abrupt Termination

Still simplified rule

$$\frac{\begin{array}{l}\Gamma \vdash \mathcal{U} \mathit{Inv} \\ \mathit{Inv},\ \texttt{exp} \vdash [\alpha']((\neg \mathit{exc} \rightarrow \mathit{Inv}) \wedge (\mathit{exc} \rightarrow [\pi\ \texttt{throw theExc;}\ \omega]\varphi)) \\ \mathit{Inv},\ \neg\texttt{exp} \vdash [\pi\ \omega]\varphi\end{array}}{\Gamma \vdash \mathcal{U}[\pi\ \texttt{while(exp)}\ \{\ \alpha\ \}\ \omega]\varphi}$$

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We omit the anonymous updates and consider exceptions as the only source for abrupt terminations.

Improved Invariant Rule—Motivation

Example

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int getMin(int [] a) {
  int m=a[0];
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  while (i<a.length) {
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- requires precodition φ_{inv}

  $$\varphi_{inv} \rightarrow [\mathrm{getMin(a)}](\varphi_{min} \wedge \varphi_{inv})$$

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A modifier set for methods has the following semantics:
After execution of the method, every location in the modifier set has the same value as in the beginning.
Analogously the same holds for loops.
In particular this means that the value of a location within the execution of the method, resp. loop body, can differ from the value in the beginning and end.

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Improved Invariant Rule—Demo

Time for a Demo

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  - enumerate what does not change
- Make proof process more efficient

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anon. update $\mathcal{V}_{Mod} = \{0 \leq x < a.length \, ? \, a[x] := c_a[x], \; i := c_i\}$